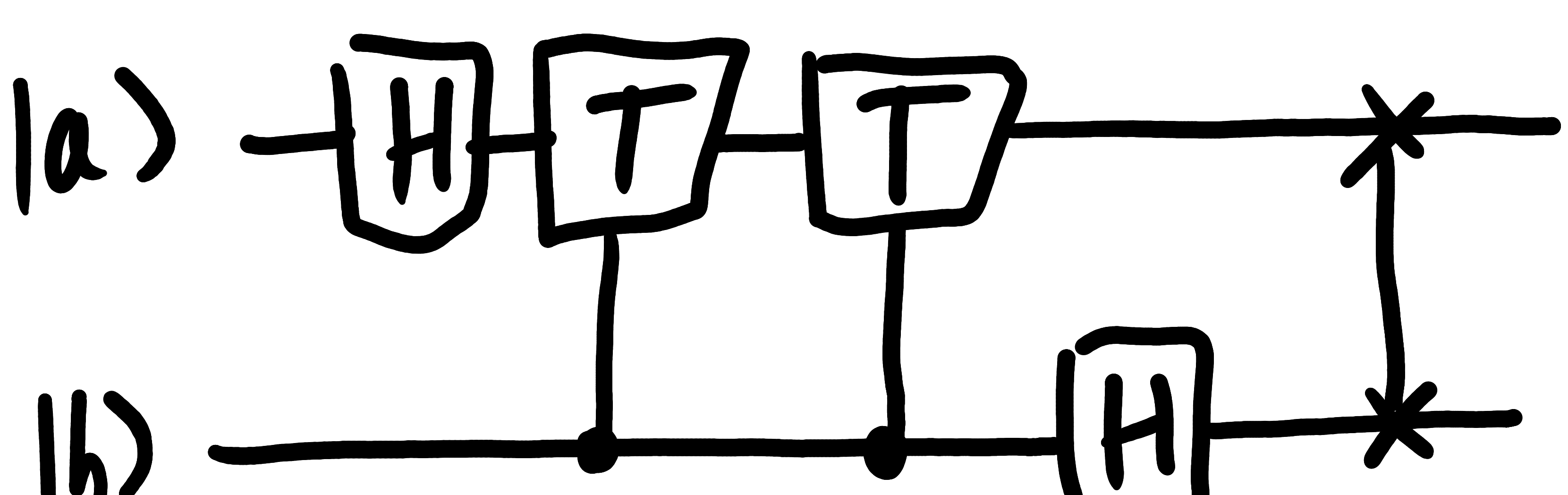
Ex. 1 (QFT)

$$\boxed{F_2} \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \equiv \boxed{H}$$

$$\boxed{F_4} \equiv \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$



$$|a\rangle|b\rangle \to \frac{|0\rangle + (-1)^a |a\rangle}{\sqrt{2}} |b\rangle$$

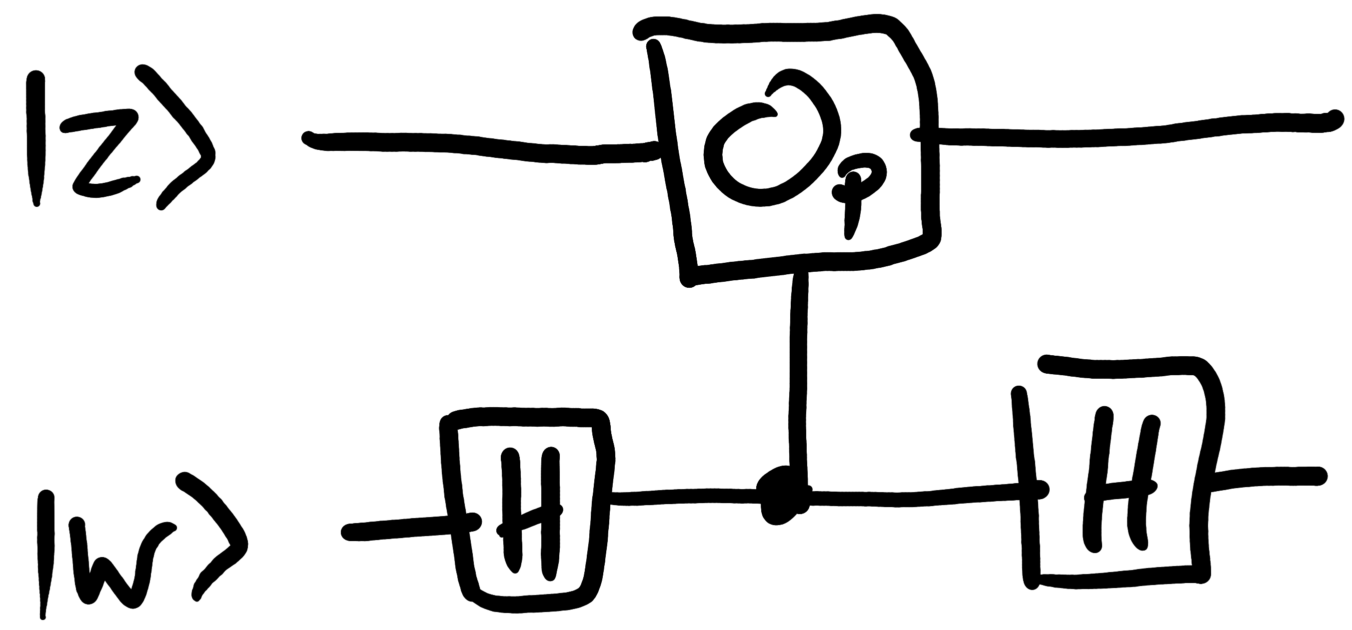
$$\to \frac{|0\rangle + (-1)^a i^b |1\rangle}{\sqrt{2}} |b\rangle$$

$$\to \frac{|0\rangle + (-1)^a i^b |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + (-1)^b |1\rangle}{\sqrt{2}}$$

$$\to \frac{|0\rangle + (-1)^b |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + (-1)^a i^b |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{2}\Big(|00\rangle + (-1)^a i^b |01\rangle + (-1)^b |10\rangle + (-1)^{a+b} i^b |11\rangle \Big) = F_4 |a\rangle|b\rangle$$

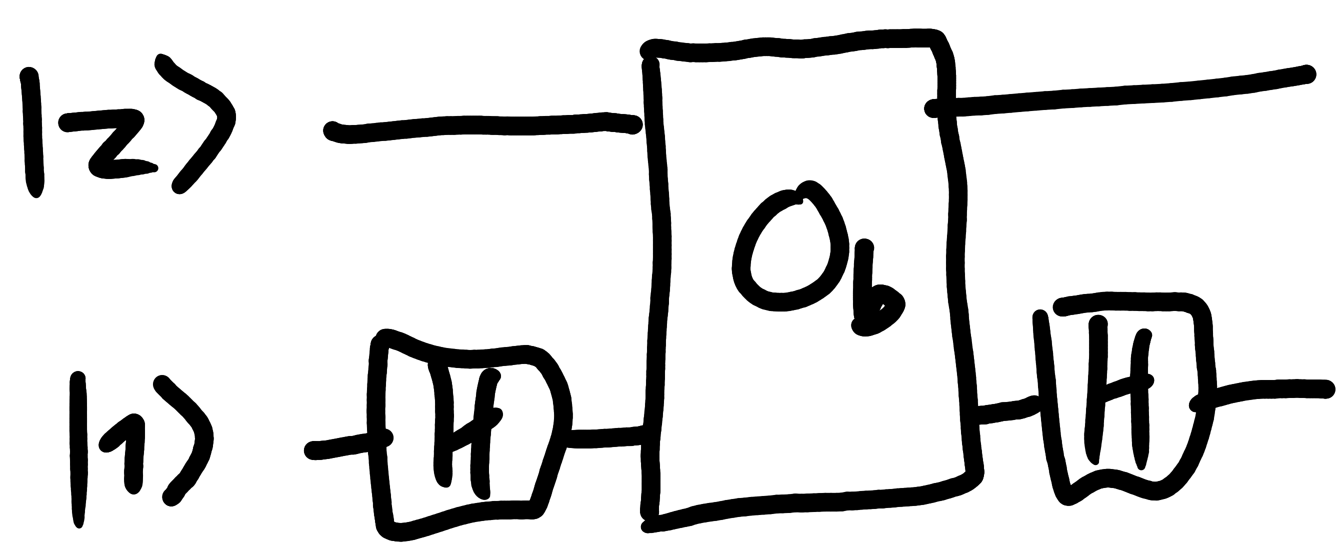
Ex. 2: oracles



$$|z\rangle|w\rangle \xrightarrow{H} |z\rangle \frac{|0\rangle + (-1)^{w}|1\rangle}{\sqrt{2}}$$

$$\xrightarrow{O_p} \frac{|z\rangle|0\rangle + (-1)^{w+f(z)}|z\rangle|1\rangle}{\sqrt{2}} \qquad = (-1)^{w\oplus f(z)}$$

$$\xrightarrow{H} |z\rangle|w\oplus f(z)\rangle = O_b\,|z\rangle|w\rangle$$



$$|z\rangle|1\rangle \xrightarrow{H} |z\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

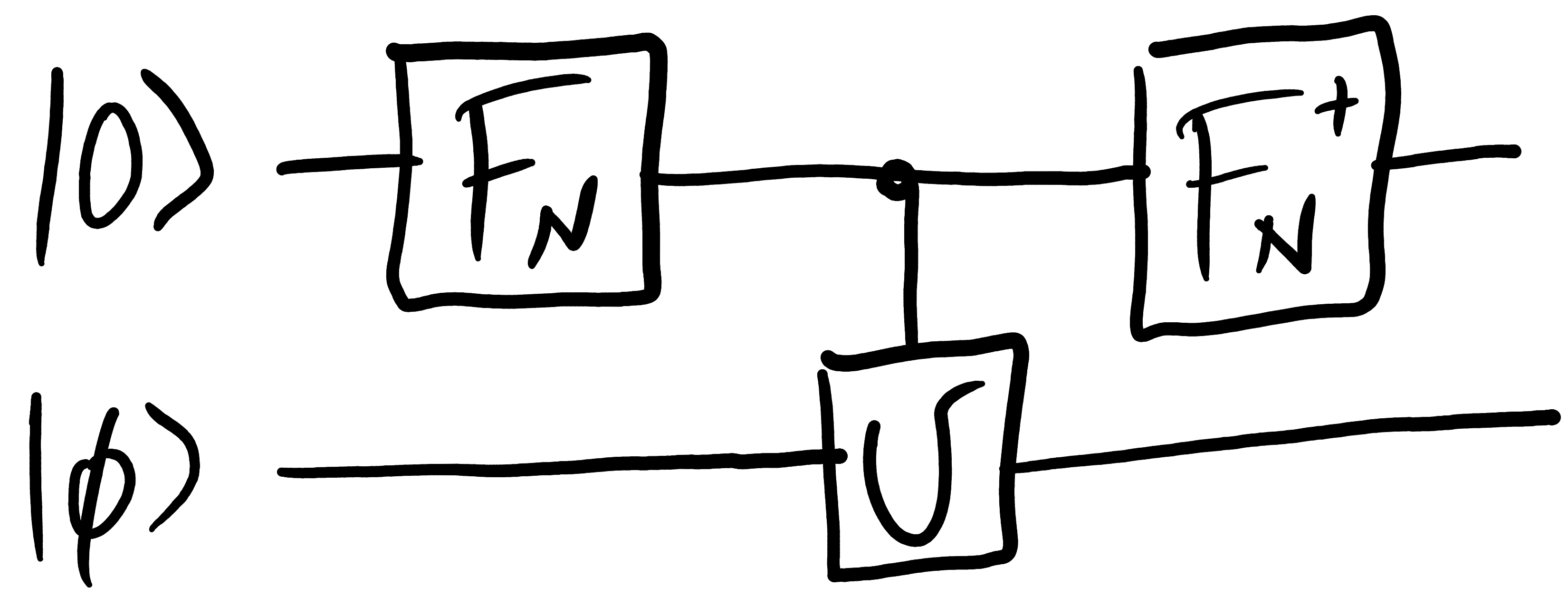
$$\xrightarrow{O_b} |x\rangle \frac{|f(x)\rangle - |1\oplus f(x)\rangle}{\sqrt{2}} = (-1)^{f(x)}|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\xrightarrow{H} (-1)^{f(x)}|x\rangle|1\rangle = \left(O_p|x\rangle\right)|1\rangle$$

Ex. 3: phase estimation

$U,\ |\phi\rangle$ s.t. $U|\phi\rangle = e^{2\pi i\theta}|\phi\rangle$ for $\phi \in [0,1)$

assume $N\theta \in \mathbb{N}$ for $N = 2^n$.



$$|0\rangle|\phi\rangle \xrightarrow{F_N} \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle|\phi\rangle$$

$$\xrightarrow{cU} \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle U^k|\phi\rangle$$

$$= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi i\theta k}|k\rangle|\phi\rangle = \left(\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \overbrace{e^{\frac{2\pi}{N}ik(\theta N)}}^{\omega_N^{k(\theta N)}}|k\rangle\right)|\phi\rangle$$

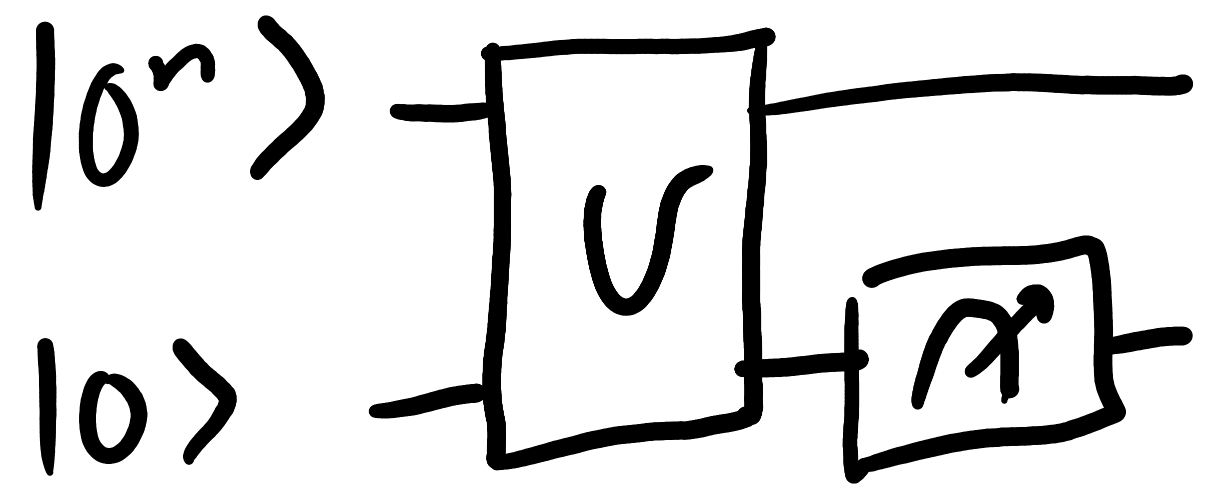
$$\xrightarrow{F_N^+} |\theta N\rangle|\phi\rangle$$

↳ learn θ

Ex. 4: amplitude estimation

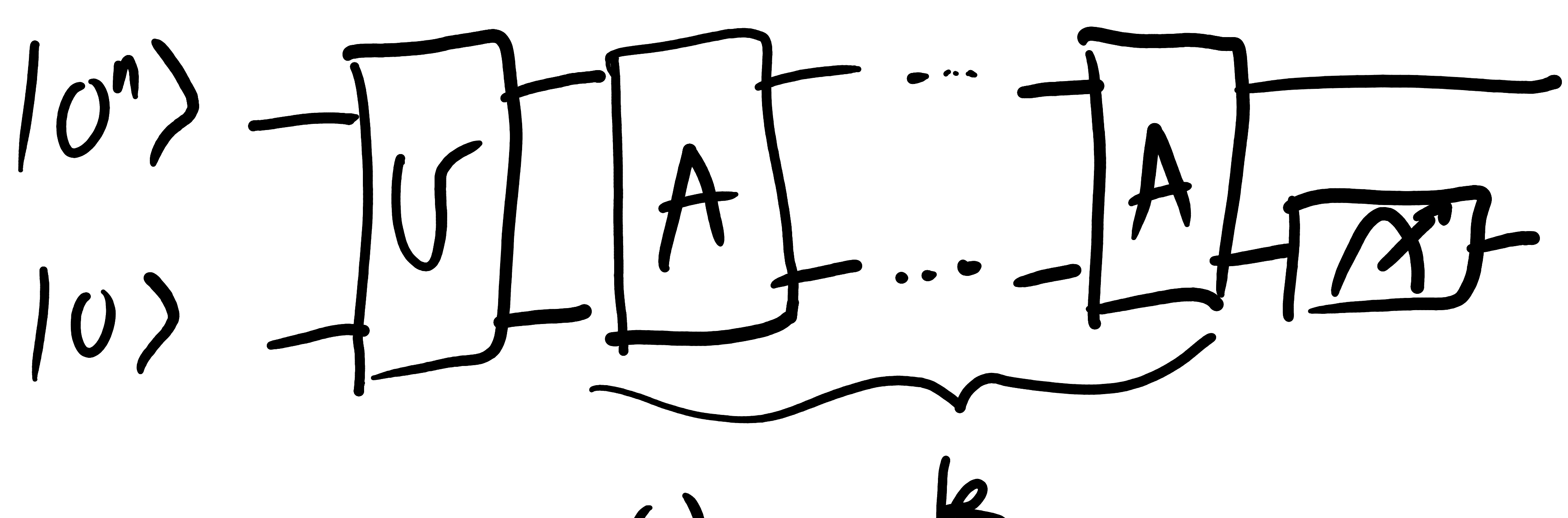
$$U|0^n\rangle|0\rangle = |\psi\rangle = \sqrt{p}\,|\psi_1\rangle|1\rangle + \sqrt{1-p}\,|\psi_0\rangle|0\rangle$$

"marked state"



$|\psi_1\rangle|1\rangle$ w. prob. p

$|\psi_0\rangle|0\rangle$ w. prob. $1-p$



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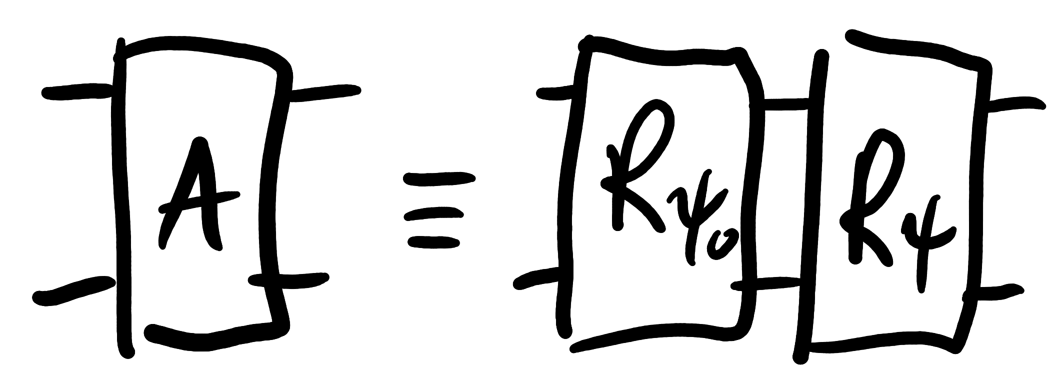


$$|0^n\rangle|0\rangle \to \overset{\sin(\theta)}{\overset{\shortparallel}{\sqrt{p}}}\,|\psi_{1,1}\rangle + \sqrt{1-p}\,|\psi_{0,0}\rangle$$

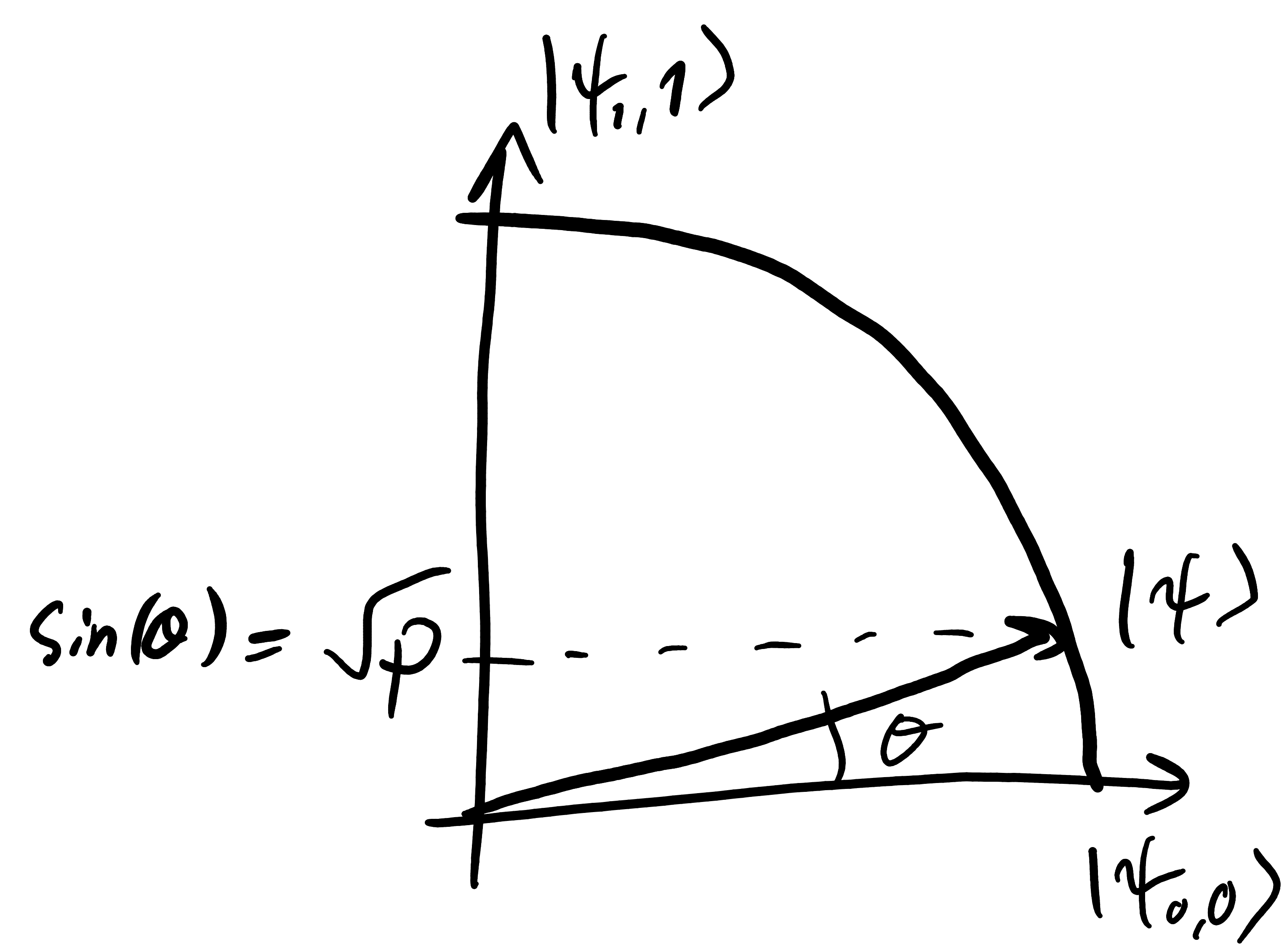
$$\xrightarrow{R_{\psi_0}} -\sqrt{p}\,|\psi_{1,1}\rangle + \sqrt{1-p}\,|\psi_{0,0}\rangle$$

$$\xrightarrow{R_{\psi}} \sin(3\theta)|\psi_{1,1}\rangle + \cos(3\theta)|\psi_{0,0}\rangle$$

$$\vdots$$

$$\xrightarrow{A} \sin((1+2k)\theta)|\psi_{1,1}\rangle + \cos((1+2k)\theta)|\psi_{0,0}\rangle$$

$$\rightarrow |\psi_{1,1}\rangle \text{ w. prob. } \sin((1+2k)\theta)^2$$



Ex. 5: approximate counting

Check that $\left\{ |\psi_\pm\rangle = \frac{|\psi_1\rangle \pm i|\psi_0\rangle}{\sqrt{2}}, \lambda_\pm = e^{\pm 2i\theta} \right\}$ are eigenpairs of A.

with θ s.t. $\sin(\theta) = \sqrt{p}$

$\hookrightarrow$ use that $A|\psi_1\rangle = \cos(2\theta)|\psi_1\rangle - \sin(2\theta)|\psi_0\rangle$

$A|\psi_0\rangle = \sin(2\theta)|\psi_1\rangle + \cos(2\theta)|\psi_0\rangle.$

Use QPE on $|\psi\rangle = -\frac{i}{\sqrt{2}}\left(e^{i\theta}|\psi_+\rangle - e^{-i\theta}|\psi_-\rangle\right)$ to estimate p.

assume $N\theta \in \mathbb{N}$

$\hookrightarrow |\psi\rangle|0\rangle \xrightarrow{QPE} \frac{-i}{\sqrt{2}}\left(e^{i\theta}|\psi_+\rangle|N\theta\rangle - e^{-i\theta}|\psi_-\rangle|-N\theta\rangle\right)$

$\xrightarrow{\nearrow} \pm N\theta.$