AQAlg: Advanced Quantum Algorithms

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Lecture 8: Quantum walk search

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1 Random walks: mixing time and hitting time

We consider a simple (undirected, unweighted) and d-regular graph G = (V, E) with |V| = n vertices. A random walk on G starts from some initial vertex (sampled from a distribution p_0 over V), and at every timestep hops uniformly at random to one of its d neighboring vertices. We can describe the probability distribution after t steps using a stochastic transition matrix P where $P_{x,y} = 1/d$ if $(x, y) \in E$ and $P_{x,y} = 0$ otherwise. After t steps the random walk distribution is

$$p_t = P^t p_0.$$

If the graph G is connected then P has a unique stationary distribution π such that $P\pi = \pi$, and moreover this is the unique eigenvalue-1 eigenvector of P.

Exercise 1. Using that G is regular, argue that π is the uniform distribution.

If in addition G is not bipartite, then p_t converges to π as $t \to \infty$, irrespective of the initial distribution p_0 . The time it takes to get close to π is quantified by the *mixing time*,

$$\mathrm{MT}(\epsilon) = \min\{t \mid \|P^t p_0 - \pi\|_1 \le \epsilon, \,\forall p_0\}.$$

We can relate the mixing time to the spectral gap δ of the transition matrix P. If we order the (real) eigenvalues of P as $1 = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge -1$, then the spectral gap is defined as

$$\delta = 1 - \max\{|\lambda_2|, |\lambda_n|\}.$$

The graph is connected and nonbipartite (i.e., has a finite mixing time) if and only if $\delta > 0$. In fact, it holds that

$$\operatorname{MT}(\epsilon) \in O\left(\frac{1}{\delta}\log\frac{n}{\epsilon}\right).$$

A different quantity of interest is the random walk *hitting time* HT(M), defined with respect to some subset of "marked" elements $M \subseteq V$ of size |M| = m. We define it as the expected number of steps of a random walk, starting from the stationary distribution $p_0 = \pi$, until it hits an element of M. The hitting time can also be bounded in terms of the spectral gap:

$$\operatorname{HT}(M) \in O\left(\frac{1}{\delta}\frac{n}{m}\right).$$

2 Quantum walks

While a random walk is defined over the vertices of a graph, a quantum walk is defined over its edges. Specifically, the state space of a quantum walk is spanned by states of the form $|x, y\rangle$ for $(x, y) \in E$. You can think about the first register as containing the "current" state x, while the

second register contains the "next" state y. In that sense, we could implement a "step" of the quantum walk through the shift operator S defined by

$$S |x, y\rangle = |y, x\rangle.$$

Instead of trivially repeating this, we alternate a step with a "coin toss" that mixes up the next state. We define it using so-called *star states* $|\psi_x\rangle$ for $x \in V$, defined as

$$|\psi_x\rangle = \frac{1}{\sqrt{d}} \sum_{(x,y)\in E} |x,y\rangle.$$

We can define a unitary coin toss operator C(P) based on these star states. Specifically, the coin toss implements a reflection around the star states:

$$C(P) = 2\left(\sum_{x \in V} |\psi_x\rangle \langle \psi_x|\right) - I.$$

The quantum walk operator W(P) is now described as

$$W(P) = S \cdot C(P).$$



Figure 1: (1) A basis state $|x, y\rangle$ is identified with the (directed) edge (x, y). (m) The coin toss C(P) maps an initial state $|x, y\rangle$ to a superposition of outgoing edges. (r) The shift S maps a state $|x, y\rangle$, localized on node x, to a state $|y, x\rangle$, localized on node y.

Exercise 2. Show that the following quantum state is a stationary state of W(P):

$$|\pi\rangle = \frac{1}{\sqrt{n}} \sum_{x \in V} |\psi_x\rangle = \frac{1}{\sqrt{nd}} \sum_{(x,y) \in E} |x,y\rangle.$$

This shows that the QW operator has an invariant eigenstate $|\pi\rangle$ that is a quantum version of the RW stationary distribution π . Similarly to the RW spectral gap δ , we can define the gap $\Delta > 0$ of the QW operator as the smallest nonzero phase such that $e^{i2\pi\Delta}$ is an eigenvalue of W. The following lemma shows that the quantum gap is quadratically larger than the random walk gap.

Lemma 1 ([Sze04]). If the Markov chain P has spectral gap δ , then the quantum walk operator W(P) has gap

$$\Delta \in \Omega(\sqrt{\delta}).$$

3 Quantum walk search

Consider again a graph G = (V, E) with |V| = n nodes and spectral gap δ . Let $M \subseteq V$ denote a subset of marked nodes of size |M| = m. An implementation of Grover search corresponds to the following:

1. Set up the stationary state

$$|\pi\rangle = \frac{1}{\sqrt{n}} \sum_{x \in V} |\psi_x\rangle.$$

- 2. Repeat $O(\sqrt{n/m})$ times:
 - (a) Reflect around marked subspace $\operatorname{span}_{x \in M} \{ |\psi_x \rangle \}$ (i.e., apply $2 \sum_{x \in M} |\psi_x \rangle \langle \psi_x | I \rangle$.
 - (b) Reflect around stationary state $|\pi\rangle$ (i.e., apply $2 |\pi\rangle \langle \pi| I$).

The resulting state will have a constant overlap with the marked state $|\pi_M\rangle = \frac{1}{\sqrt{m}} \sum_{x \in M} |\psi_x\rangle$, so that measuring the state returns a marked element with constant probability.

The idea of quantum walk search is to implement the reflection around $|\pi\rangle$ using a quantum walk. The resulting algorithm is called the "MNRS algorithm", after Magniez-Nayak-Roland-Santha [MNRS07].

Exercise 3 (Reflecting around $|\pi\rangle$). We can use a quantum walk W(P) to reflect around the quantum state $|\pi\rangle$, i.e., implement the map

$$|\psi\rangle = \alpha |\pi\rangle + \beta |\pi^{\perp}\rangle \longrightarrow (2 |\pi\rangle \langle \pi| - I) |\psi\rangle = \alpha |\pi\rangle - \beta |\pi^{\perp}\rangle.$$

Assume that $|\psi\rangle = \alpha |\pi\rangle + \sum_{j} \beta_{j} |v_{j}\rangle$ such that $W(P) |\pi\rangle = |\pi\rangle$ and $W(P) |v_{j}\rangle = e^{i2\pi\theta_{j}} |v_{j}\rangle$ with $1/2 > |\theta_{j}| > \Delta > 0$. We call Δ the spectral gap of the quantum walk. Argue that the following circuit implements a reflection around $|\pi\rangle$ (QPE represents quantum phase estimation with respect to W(P) to precision $\Delta/2$).



How many calls does the circuit make to the quantum walk operator? Conclude that quantum walk search finds a marked element using a number of quantum walk steps in

$$O\left(\frac{1}{\sqrt{\delta}}\sqrt{\frac{n}{m}}\right).$$

References

- [MNRS07] Frédéric Magniez, Ashwin Nayak, Jérémie Roland, and Miklos Santha. Search via quantum walk. In Proceedings of the thirty-ninth annual ACM symposium on Theory of computing, pages 575–584, 2007.
- [Sze04] Mario Szegedy. Quantum speed-up of Markov chain based algorithms. In Proceedings of the 45th IEEE Symposium on Foundations of Computer Science (FOCS), pages 32–41. IEEE, 2004.