EX: Schrödinger's equation

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -iH|\psi(t)\rangle$$

↳ Verify that $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$:

$$\partial_t|\psi(t)\rangle = \partial_t e^{-iHt}|\psi(0)\rangle$$

$$= \partial_t \sum_{k=0}^{+\infty} \frac{1}{k!}(-iHt)^k|\psi(0)\rangle$$

$$= \sum_{k=1}^{+\infty} \frac{1}{k!} k(-iH)(-iHt)^k|\psi(0)\rangle$$

$$= -iH \sum_{k=0}^{+\infty} \frac{1}{k!}(-iHt)^k|\psi(0)\rangle$$

$$= -iH|\psi(t)\rangle$$

Ex:

Let $-\boxed{R_z(\theta)}- \equiv e^{-i\theta Z} = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$.

Show that

$$-\boxed{H}-\boxed{Z}-\boxed{H}- \;\equiv\; -\boxed{X}-$$

$$\hookrightarrow = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

and $e^{-i\theta X} \equiv -\boxed{H}-\boxed{R_z(\theta)}-\boxed{H}-$

$$\hookrightarrow = \sum_{k=0}^{+\infty} \frac{1}{k!}(-i\theta X)^k$$

$$= \sum_{k=0}^{+\infty} \frac{1}{k!}(-i\theta HZH)^k$$

$$= H\left(\sum_{k=0}^{+\infty} \frac{1}{k!}(-i\theta Z)^k\right)H = He^{-i\theta Z}H$$

using that $H^2 = \mathbb{1}$

EX: Lie-Trotter

A, B Hermitian, $\|A\|, \|B\| \leq 1$, $0 < \delta < 1$

Show that $e^{(A+B)\delta} = e^{A\delta} e^{B\delta} + O(\delta^2)$.

↳ Use Taylor expansion:

$$e^{(A+B)\delta} = \sum_{k=0}^{+\infty} \frac{1}{k!} ((A+B)\delta)^k$$

$$= \mathbb{1} + (A+B)\delta + O(\delta^2)$$

and $$e^{A\delta} e^{B\delta} = (\mathbb{1} + A\delta + O(\delta^2))(\mathbb{1} + B\delta + O(\delta^2))$$

$$= \mathbb{1} + (A+B)\delta + O(\delta^2).$$

EX: measuring $\langle\psi|Z|\psi\rangle$.

Using $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ we have

$$\langle\psi|Z|\psi\rangle = (\bar{\alpha}_0\langle 0| + \bar{\alpha}_1\langle 1|)\, Z\, (\alpha_0|0\rangle + \alpha_1|1\rangle)$$
$$= (\bar{\alpha}_0\langle 0| + \bar{\alpha}_1\langle 1|)(\alpha_0|0\rangle - \alpha_1|1\rangle)$$
$$= |\alpha_0|^2 - |\alpha_1|^2 = 2|\alpha_0|^2 - 1.$$

(using that $|\alpha_0|^2 + |\alpha_1|^2 = 1$

EX: measuring $\langle\psi|X|\psi\rangle$.

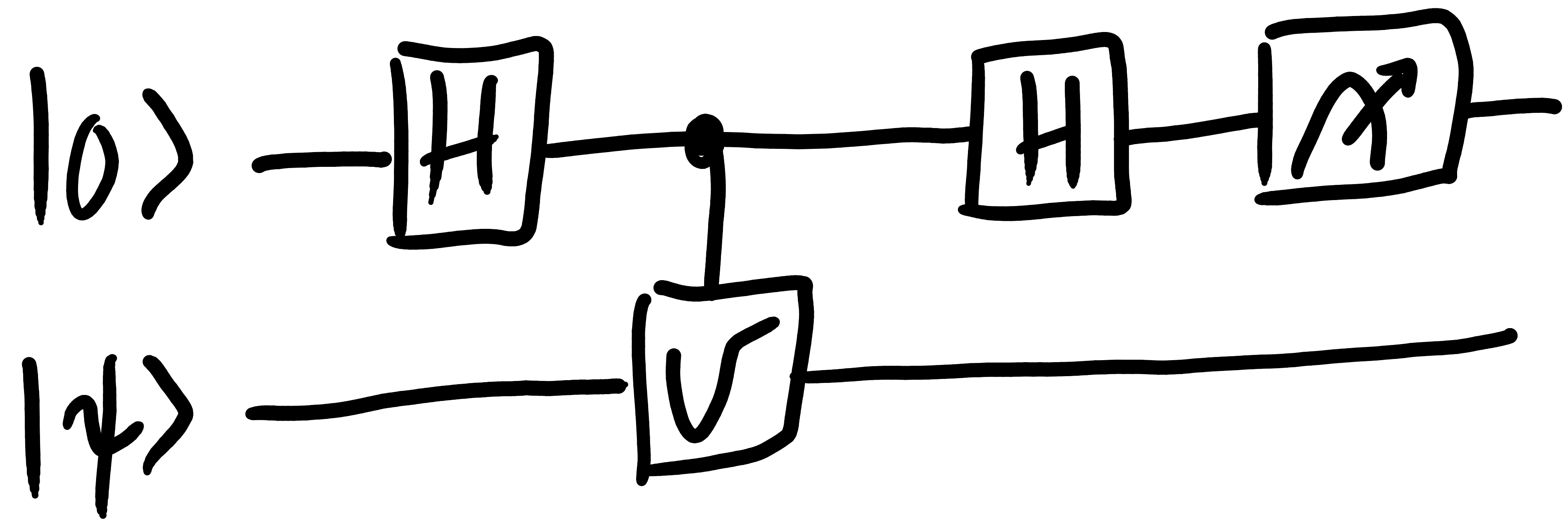
maps $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

$$\xrightarrow{H} \alpha_0\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \alpha_1\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$
$$= \left(\frac{\alpha_0 + \alpha_1}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha_0 - \alpha_1}{\sqrt{2}}\right)|1\rangle$$
$$\xrightarrow{\nearrow} \text{"0" w. prob. } \tfrac{1}{2}|\alpha_0 + \alpha_1|^2$$

and
$$\langle\psi|X|\psi\rangle = (\bar{\alpha}_0\langle 0| + \bar{\alpha}_1\langle 1|)(\alpha_0|1\rangle + \alpha_1|0\rangle)$$
$$= \bar{\alpha}_0\alpha_1 + \bar{\alpha}_1\alpha_2$$
$$= |\alpha_0 + \alpha_1|^2 - 1.$$

EX: unitary Hamiltonian

consider U s.t. $U^{\dagger}U = \mathbb{1}$, $U = U^{\dagger}$



maps $|0\rangle|\psi\rangle \xrightarrow{H} \dfrac{|0\rangle|\psi\rangle + |1\rangle|\psi\rangle}{\sqrt{2}}$

$$\xrightarrow{cU} \frac{|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle}{\sqrt{2}}$$

$$\xrightarrow{H} \frac{|0\rangle|\psi\rangle + |1\rangle|\psi\rangle + |0\rangle U|\psi\rangle - |1\rangle U|\psi\rangle}{2}$$

$$= |0\rangle\left(\frac{|\psi\rangle + U|\psi\rangle}{2}\right) + |1\rangle\left(\frac{|\psi\rangle - U|\psi\rangle}{2}\right)$$

$\xrightarrow{\nearrow}$ "0" w. prob.

$$\left\| \frac{|\psi\rangle + U|\psi\rangle}{2} \right\|^2 = \frac{1}{4}\left(\langle\psi| + \langle\psi|U^{\dagger}\right)\left(|\psi\rangle + U|\psi\rangle\right)$$

$$= \frac{1}{4}\left(1 + \langle\psi|U|\psi\rangle + \langle\psi|U^{\dagger}|\psi\rangle + \langle\psi|U^{\dagger}U|\psi\rangle\right)$$

$$= \frac{1}{2}\left(1 + \langle\psi|U|\psi\rangle\right).$$