# A Unified Framework of Quantum Walk Search


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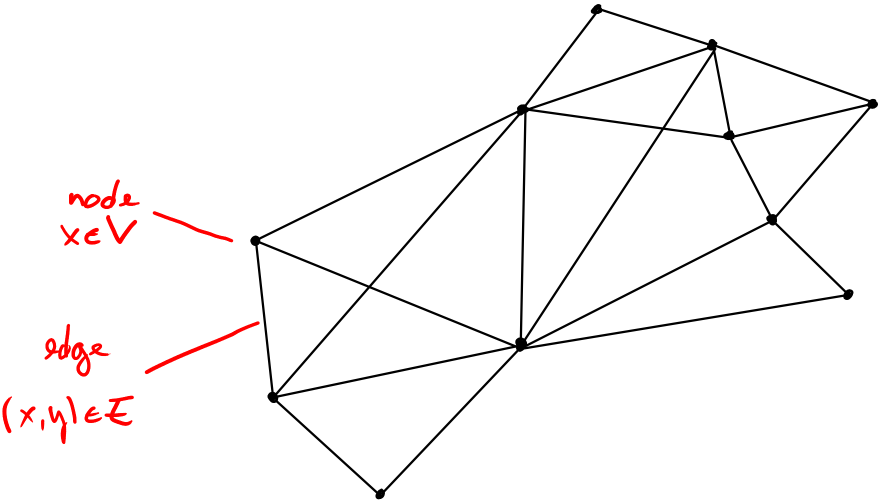
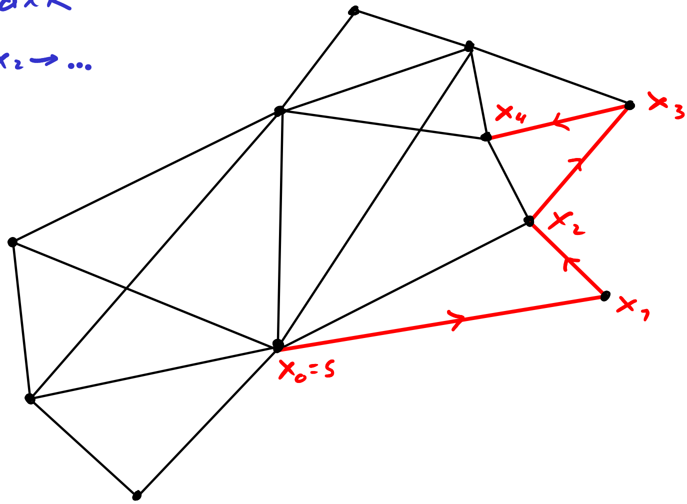
Graph $G=(V,E)$



# Random Walk

$s = X_0 \to X_1 \to X_2 \to \dots$

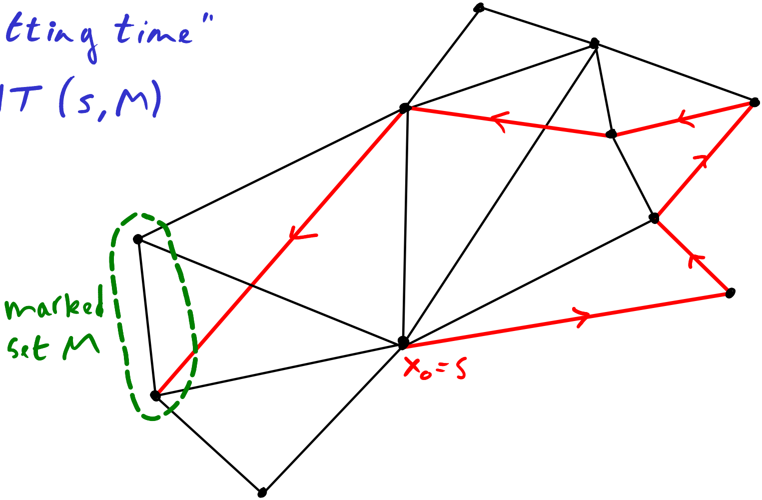


(expected)

? time for RW from s to hit set M?

= "hitting time"

$HT(s, M)$



Why are we interested in random walks, hitting times, ...?

**GREAT MATHS**

natural way to study graphs and electric networks

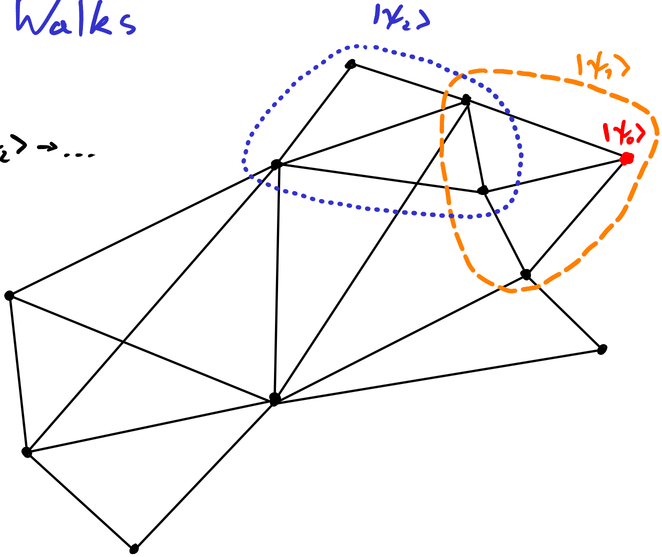
**GREAT ALGORITHMS**

building block of many algorithms

e.g. s-t connectivity in log-space

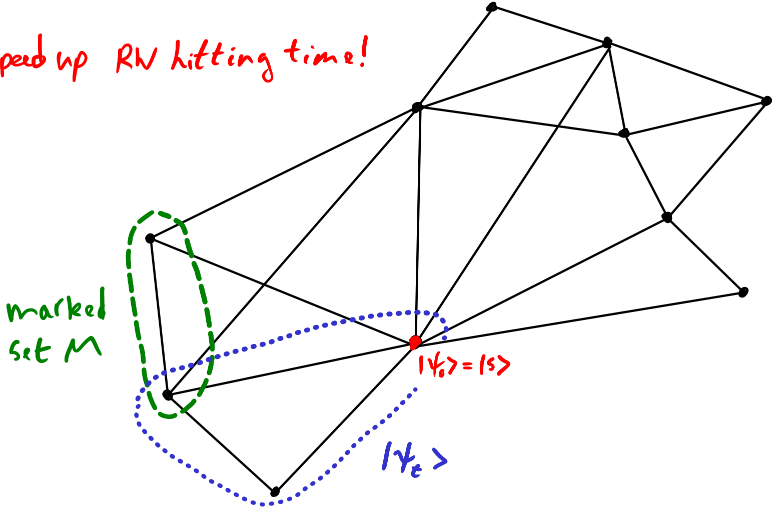
# Quantum Walks

$|\psi_0\rangle \to |\psi_1\rangle \to |\psi_2\rangle \to \ldots$



? time for QW from s to "hit" set M ?

! can speed up RW hitting time!



# Quantum walks and Hitting times

## GREAT MATHS

* natural application of Jordan's lemma (1875)
* close relation to electric network theory [Belovs 13, Piddock 19]

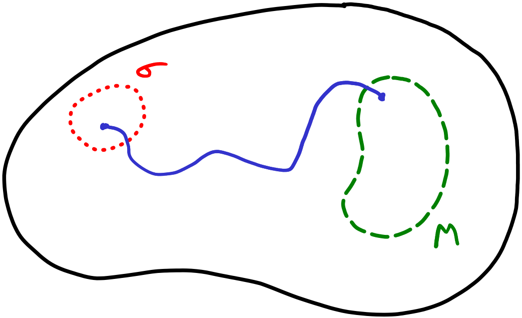
## GREAT QUANTUM ALGORITHMS

! can speed up RW hitting

* element distinctness [Ambainis 03]
* quantum backtracking [Montanaro 15]
* ...

formally,

use RW from $X_0 \sim \sigma$ to find $m \in M$ $\longleftrightarrow$ use QW from $|\psi_0\rangle = |\sigma\rangle$ to find $m \in M$



with minimal "costs":

setup cost $S$

sample $X \sim \sigma$

create $|\sigma\rangle$

update cost $U$

sample $y$ from $N_x$

map $|x\rangle \to \sum_{y \in N_x} |y\rangle$

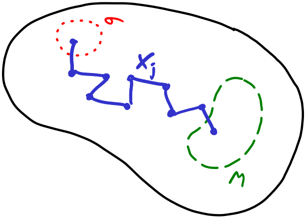
checking cost $C$

check if $z \in M$

map $|z\rangle \to \begin{cases} -|z\rangle & \text{if } z \in M \\ |z\rangle & \text{o/w} \end{cases}$

Classically,

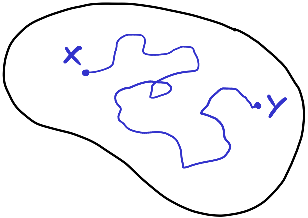
"hitting time algorithm"




$$\xrightarrow{S} \underset{\sim\sigma}{X_0} \underbrace{\xrightarrow{C+U} X_1 \xrightarrow{C+U} \cdots \xrightarrow{C+U} X_T \in M}_{HT(\sigma,M)}$$

cost $\quad S + HT(C+U)$

"mixing time algorithm"



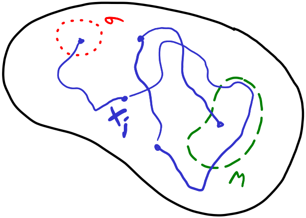

define

"spectral gap" $\delta$ s.t. $X \xrightarrow{\frac{1}{\delta} U} Y \sim \pi$

$\varepsilon = \mathbb{P}(X \in M \mid X \sim \pi)$

then $\frac{1}{\varepsilon} \leq HT \leq \frac{1}{\varepsilon \delta}$

"mixing time algorithm"




$$\xrightarrow{S} \underset{\sim\sigma}{X_0} \xrightarrow{C+\frac{1}{\delta}U} \underbrace{\underset{\sim\pi}{X_1} \xrightarrow{C+\frac{1}{\delta}U} \underset{\sim\pi}{X_2} \longrightarrow \dots \longrightarrow X_T \in M}_{\frac{1}{\varepsilon} = \frac{1}{\pi(M)}}$$

$$\text{cost} \quad S + \frac{1}{\varepsilon\delta} U + \frac{1}{\varepsilon} C$$

$$(\longleftrightarrow \quad S + \check{HT}\, U + \hat{HT}\, C)$$

# Quantum Walk Search Algorithms

≈ "Grover on Graphs"

'96 Grover Search

'03, '04 QW search speedup on lattices, Johnson graphs, hypercubes, ...

— ad hoc analysis

'04-... QW speedups for general graphs

[Szegedy '04]: if $\sigma = \pi$, possible to detect if $M \neq \emptyset$ in

$$S + \sqrt{HT}\,(U + C)$$

[MNRS '06]: if $\sigma = \pi$, possible to find $m \in M$ in

$$S + \frac{1}{\sqrt{\varepsilon\delta}} U + \frac{1}{\sqrt{\varepsilon}} C$$

[KMOR '10]: if $\sigma = \pi$ and $M = \{m\}$, possible to find $m$ in

$$S + \sqrt{HT}\,(U + C)$$

[Belovs '13]: for any $\sigma$, possible to detect if $M \neq \emptyset$ in

$$S + \underbrace{\sqrt{Rm}}_{= HT \text{ if } \sigma = \pi}(U + C)$$

[DH '17]: if $\sigma = \pi$ and $M = \{m\}$, possible to find $m$ in

$$S + \sqrt{HT}\, U + \frac{1}{\sqrt{\varepsilon}} C$$

[AGJK '19]: if $\sigma = \pi$, possible to find $m \in M$ in

$$S + \sqrt{HT}\,(U + C)$$

| | | | | | |
|---|---|---|---|---|---|
| [S'09], [KMOR'16], [AGJK'19] | finds | in | $S+\sqrt{HT}(U+C)_{(1)}$ | if | $\sigma=\pi$ |
| [MNRS'06] | finds | in | $S+\frac{1}{\sqrt{\varepsilon\delta}}U+\frac{1}{\sqrt{\varepsilon}}C_{(2)}$ | if | $\sigma=\pi$ |
| [Belovs'13] | detects | in | $S+\sqrt{Rm}(U+C)_{(3)}$ | for | any $\sigma$ |
| [DH'17] | finds | in | $S+\sqrt{HT}U+\frac{1}{\sqrt{\varepsilon}}C_{(4)}$ | if | $\sigma=\pi$ and $M=\{m\}$ |

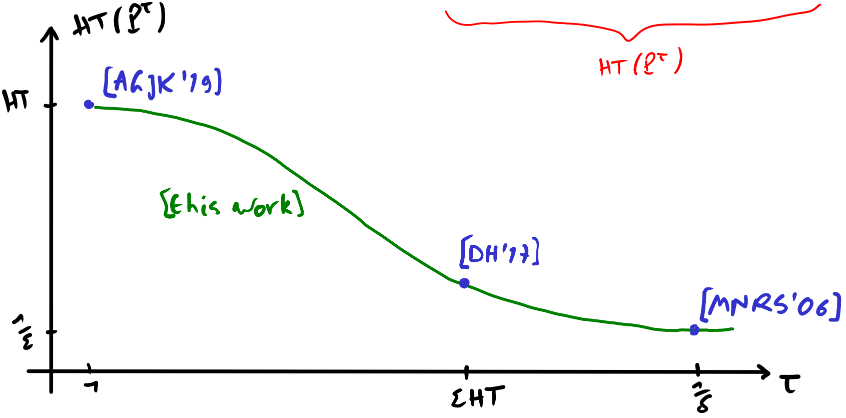
[this work]: finds in

$$S+\sqrt{Rm(\rho^{\tau})}\left(\sqrt{\varepsilon}\,U+C\right)=\begin{cases}(1) & \text{if } \tau=1,\ \sigma=\pi\\ (2) & \text{if } \tau=\frac{1}{\delta},\ \sigma=\pi\\ (3) & \text{if } \tau=1\\ (4) & \text{if } \tau=\varepsilon HT,\ \sigma=\pi,\ M=\{m\}\end{cases}$$

for any $\sigma$

* if $\sigma = \pi$, then $RM(P^\tau) = HT(P^\tau)$ :

$$\underbrace{X_0 \xrightarrow{\tau U} X_1 \xrightarrow{\tau U} \cdots \to X_T \in M}_{HT(P^\tau)}$$



* if $\tau = 1$, then $Rm(P^\tau) = Rm$:

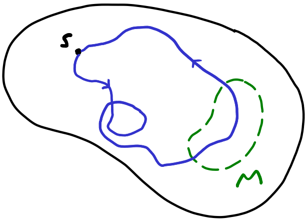
[Belovs'13] detects in $S + \sqrt{Rm}(U+C)$ for any $\sigma$

→ [this work] finds in $S + \sqrt{Rm}(U+C)$ for any $\sigma$

(+ independently [Piddock'19])

e.g., if $|\sigma\rangle = |s\rangle$,

then $Rm$ = "commute time"




NOTE: when $\sqrt{Rm} \gg HT$, no quantum speedup

Additional results:

* if $\tau = 1$, only need QWs (no phase estimation, QFT)

  $\rightsquigarrow$ answers open question of [AGJK'19]

* Monte Carlo type bounds

# Proof Ideas

# PROOF IDEAS

* finding in $\sqrt{Rm}$ QW steps

# Quantum Fast-Forwarding [AS'18]

classical RW operator

maps

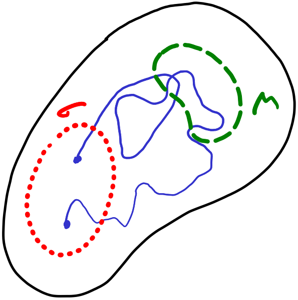
$$|\psi\rangle \longrightarrow P^t|\psi\rangle|0\rangle \quad \left(+\,|\Gamma\rangle|1\rangle\right)$$

using $O(\sqrt{t})$ QW steps.

## Lemma:

If RW "commutes" between $\sigma$ and $M$ in $O(t)$ steps, then

$$\|\Pi_M P^t |\sigma\rangle\|^2 \in \Omega(1)$$




proof: using "interpolated walks" [KMOR'16]
and [AGJK'19]

## Corollary:

Find $m \in M$ in $\sqrt{\sigma\text{-}M \text{ commute time}}$ QW steps

## Final Claim: (known for $\sigma = \langle s \rangle$ or $\sigma = \pi$)

$$\sigma\text{-}M \text{ commute time} \approx RM$$

combinatorial quantity

electric quantity

proof: using electric networks theory

## Corollary:

Find $m \in M$ in $\sqrt{RM}$ QW steps

# PROOF IDEAS

* finding in $\sqrt{Rm}$ QW steps
* avoiding QFF

our QW algorithm

$$|\sigma\rangle|0\rangle \xrightarrow{QFF} P^t|\sigma\rangle|0\rangle + |\Gamma\rangle$$

$\longrightarrow$ measure 1st register

returns $m \in M$ with high probability

"inside" the QFF box:

$$
\left.
\begin{aligned}
|\sigma\rangle|0\rangle \;&\xrightarrow{I\otimes V}\; \sum_{\ell=0}^{\sqrt{t}} \sqrt{\alpha_\ell}\, |\sigma\rangle|\ell\rangle \\
&\xrightarrow{c\text{-}W}\; \sum_{\ell=0}^{\sqrt{t}} \sqrt{\alpha_\ell}\, W^\ell|\sigma\rangle|\ell\rangle \\
&\xrightarrow{I\otimes V^\dagger}\; \underbrace{\sum_{\ell=0}^{\sqrt{t}} \alpha_\ell\, W^\ell|\sigma\rangle|0\rangle}_{\approx P^t|\sigma\rangle} \;+\; |\Gamma\rangle
\end{aligned}
\right\}\; \text{QFF}
$$

$\longrightarrow$ measure 1st register

"inside" the QFF box:

$$
\left.
\begin{aligned}
|\sigma\rangle|0\rangle &\xrightarrow{I\otimes V} \sum_{\ell=0}^{\sqrt{t}} \sqrt{\alpha_\ell}\, |\sigma\rangle|\ell\rangle \\
&\xrightarrow{c\text{-}W} \sum_{\ell=0}^{\sqrt{t}} \sqrt{\alpha_\ell}\, W^\ell|\sigma\rangle|\ell\rangle \\
&\xrightarrow{I\otimes V^\dagger} \underbrace{\sum_{\ell=0}^{\sqrt{t}} \alpha_\ell W^\ell|\sigma\rangle|0\rangle}_{\approx P^t|\sigma\rangle} + |\Gamma\rangle
\end{aligned}
\right\} \text{QFF}
$$

commute

$$\longrightarrow \quad \text{measure } 1^{\text{st}} \text{ register}$$

"inside" the QFF box:

$$|\sigma\rangle|0\rangle \xrightarrow{I\otimes V} \sum_{\ell=0}^{\sqrt{t}} \sqrt{\alpha_\ell}\, |\sigma\rangle|\ell\rangle$$

$$\xrightarrow{c\text{-}W} \sum_{\ell=0}^{\sqrt{t}} \sqrt{\alpha_\ell}\, W^\ell|\sigma\rangle|\ell\rangle$$

$\longrightarrow$ measure 1st register

after tracing out 2nd register

$\downarrow$

$W^\ell|\psi\rangle$ with probability $\alpha_\ell$

equivalent algorithm:

pick $0 \leq \ell \leq \sqrt{t}$ with probability $\alpha_\ell$

apply $\ell$ QW steps $|\sigma\rangle \to W^\ell |\sigma\rangle$

measure $W^\ell |\sigma\rangle$