ELFS, TREES AND QUANTUM WALKS

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Sublinear algorithms, Bernoulli center (EPFL), December ’22

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simple graph $G = (V, E)$

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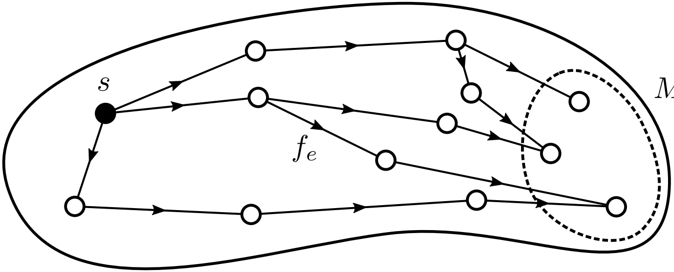
unit electric flow f from source s to sink M



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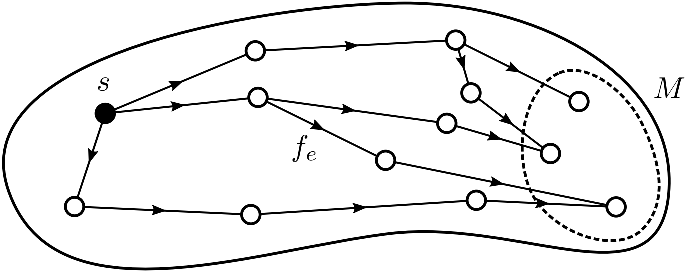
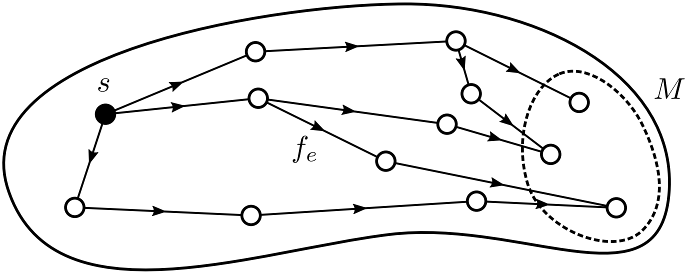


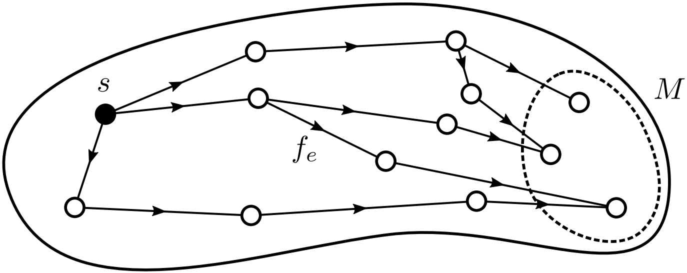
effective resistance $R = \sum_e f_e^2$

Electric flow sampling (elfs) process:



Electric flow sampling (elfs) process: repeat

1. sample edge e with probability $p_e = f_e^2/R$



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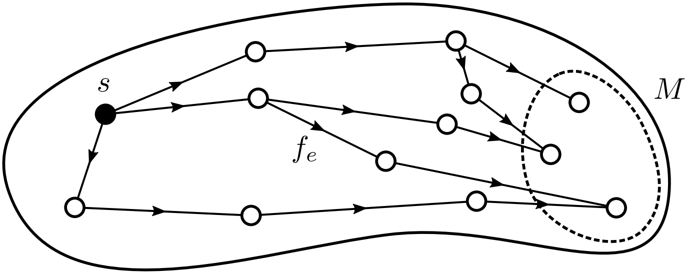

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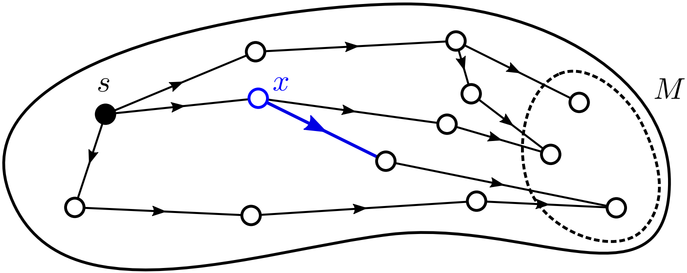

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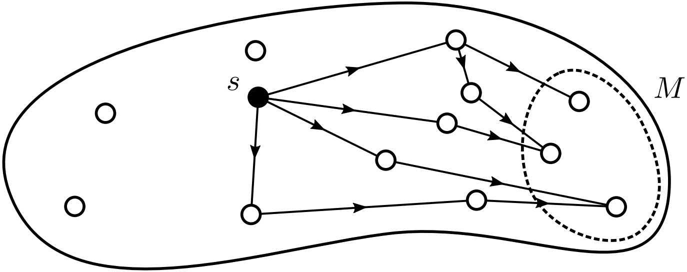

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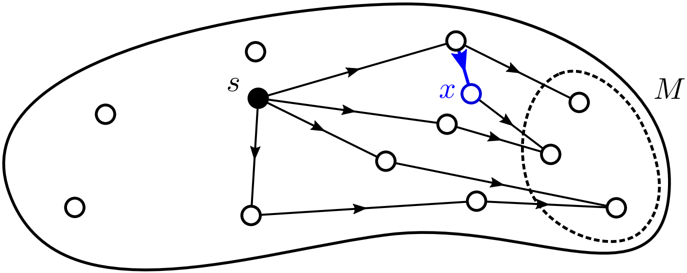

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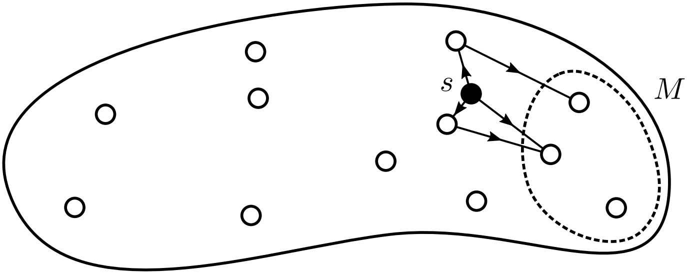

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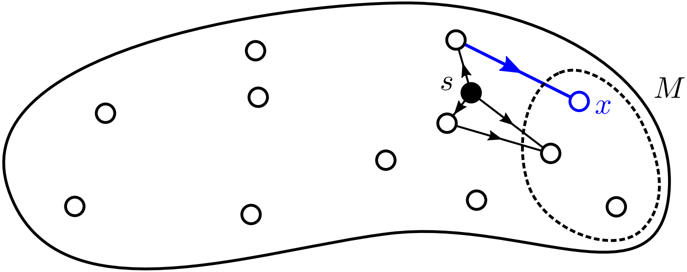

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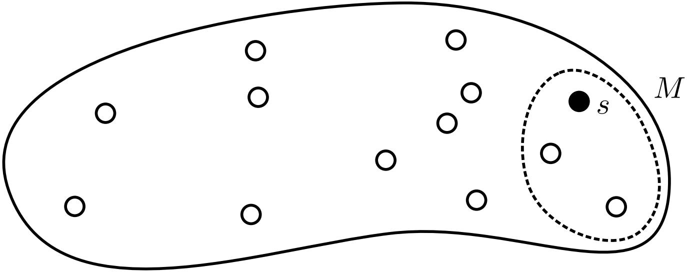

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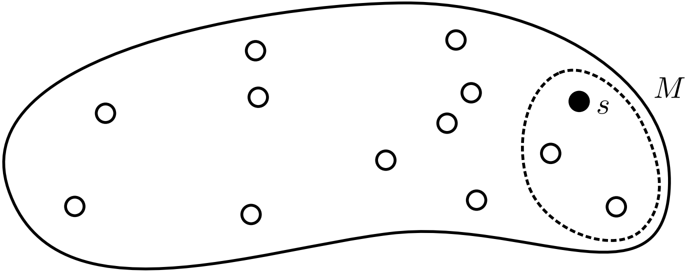

process ends when step 2. picks sink vertex

Results

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- **elfs**

coupling with random walk

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- **trees**

electric hitting time is $O(\log n)$

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coupling with random walk

- **trees**

electric hitting time is $O(\log n)$

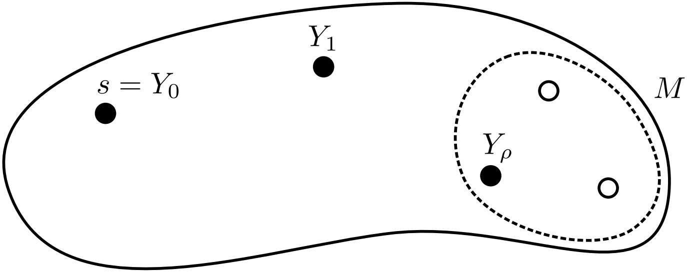
- **quantum walks**

quantum algorithm for elfs

elfs process describes Markov chain

$$\{Y_0 = s, Y_1, Y_2, \ldots, Y_\rho \in M\}$$

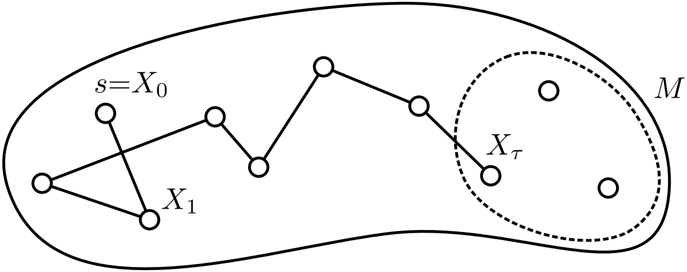
with ρ the “electric hitting time”



random walk

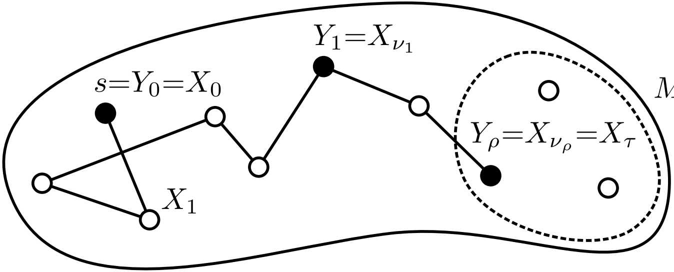
$$\{X_0 = s, X_1, X_2, \ldots, X_\tau \in M\}$$

with τ the random walk hitting time



lemma: exists stopping times $0 < \nu_1 < \cdots < \nu_\rho = \tau$ such that

$$\{Y_0 = X_0, \quad Y_1 = X_{\nu_1}, \quad \ldots, \quad Y_\rho = X_{\nu_\rho} = X_\tau\}$$



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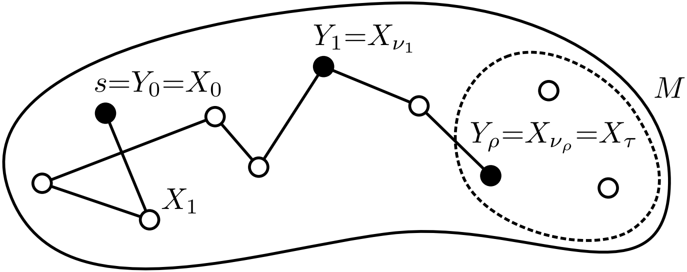

i.e., exists coupling between random walk and elfs process

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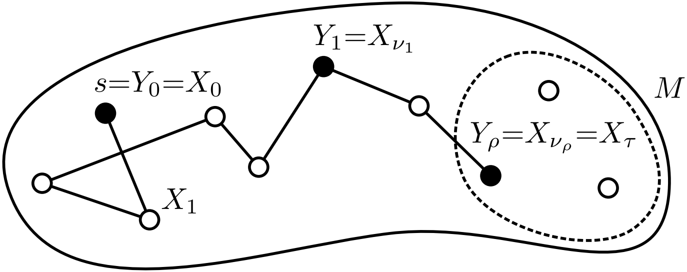

corollary: random walk and elfs process same arrival distribution

$$\Pr(Y_\rho = x) = \Pr(X_\tau = x) = \sum_y f_{y,x}, \qquad x \in M$$

stopping rule ν from s: when visiting vertex x, stop with probability[1]

$$p_x = \frac{\sum_y (v_x - v_y)^2}{v_x d_x + \sum_y (v_x - v_y)^2}$$

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e.g., $p_x = 1$ for $x \in M$

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expected stopping time $\mathbb{E}[\nu]$**?**

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using $v_x \leq v_s = R$:
$$\begin{aligned} 1 &\leq Rd_s \quad \left(= v_s d_s = 1/\Pr_s(\tau_M < \tau_s^+)\right) \\ &\leq \mathrm{ET}_s \\ &\leq \mathrm{HT}_s \quad \left(= \sum_x v_x d_x\right) \end{aligned}$$

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lemma': $\mathrm{ET}_s =$ "escape time" from s to M

$$= \mathbb{E}\{1 + \max\{t \mid X_t = s\}\}$$

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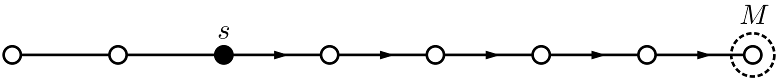

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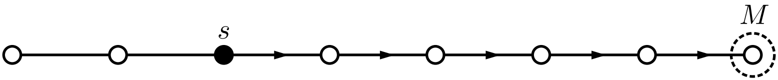

$$\mathrm{EHT}_s \sim \log n$$

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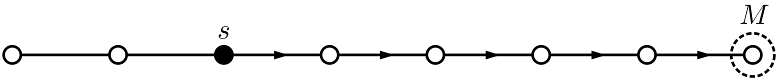

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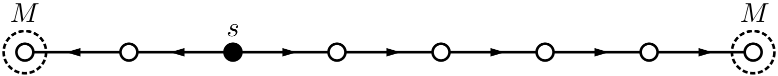

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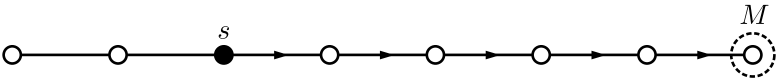

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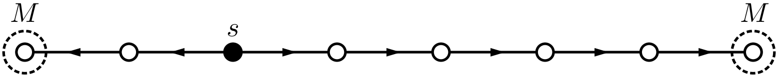

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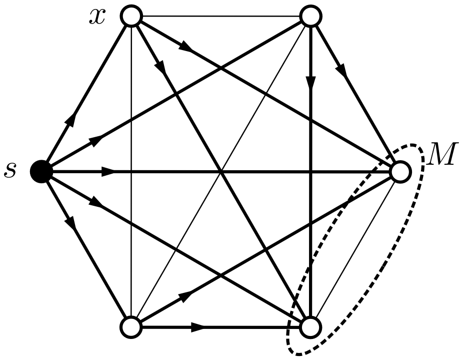

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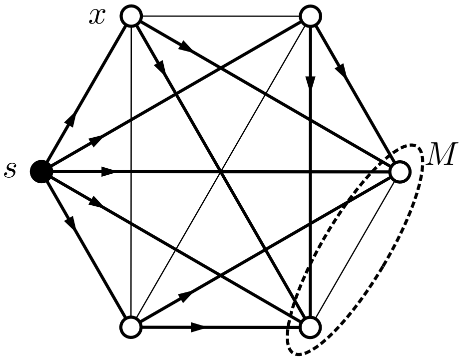

$$\mathrm{EHT}_s \sim \min\{|M|, n/|M|\}$$

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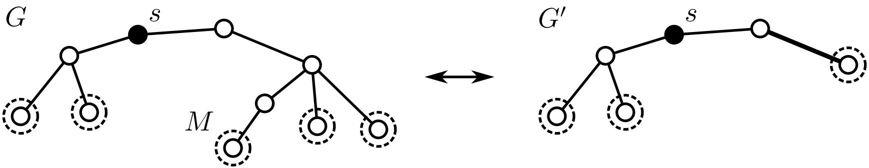
- divide via Schur complement for elfs process on trees



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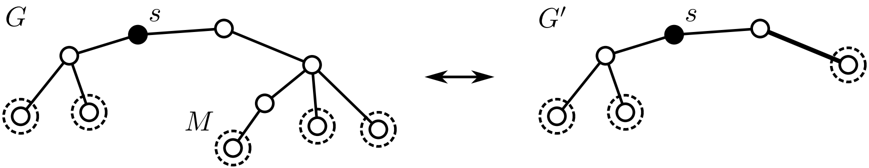

- conquer via bottoms-up:

steps below vertex = f(# steps below children)

ELFS, TREES AND **QUANTUM WALKS**

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algorithmic tool in quantum computing

(element distinctness, search, Monte Carlo)

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measuring $|f\rangle$ yields electric flow sample let $\mathrm{Ref}_{+} =$ reflection around $\mathrm{span}_{x,y}\{|x,y\rangle + |y,x\rangle\}$

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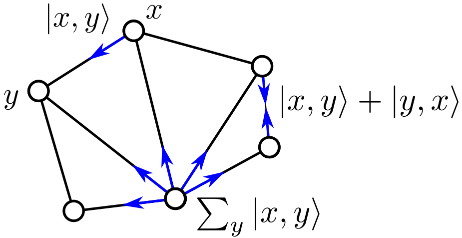

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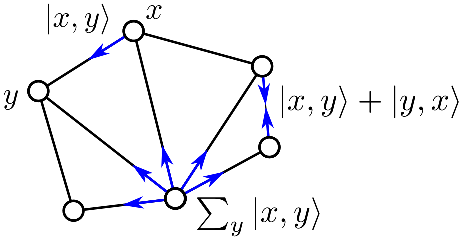

quantum walk operator

$$W = \mathrm{Ref}_+\mathrm{Ref}_*$$

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! electric flow state $|f\rangle$
orthogonal to all closed flows

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(for s-t flow)

1. start from $|s, t\rangle$

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(ignoring errors in remainder)

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many applications (to explore)
(sampling RSTs, semi-supervised learning, linear system solving)

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(quantum) algorithms for **L**x=b

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quantum/probabilistic solution	**explicit solution**

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[Apers-de Wolf, FOCS'20]

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classical in $\mathrm{poly}'(\kappa)$ via random walks / Monte Carlo	classical in $\widetilde{O}(m)$ [Spielman-Teng, STOC'04]

$\rightarrow$ **polynomial speedups**

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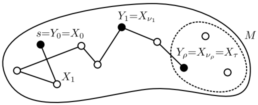
$\rightarrow$ **no speedups?**

Summary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X_1
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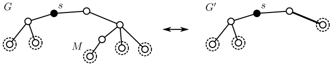
G
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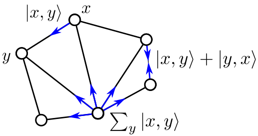
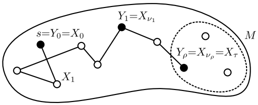
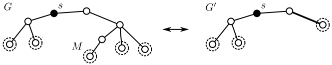
M
G'
s

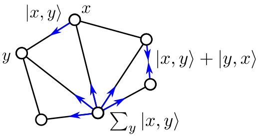

$|x, y\rangle$
x
y
$|x, y\rangle + |y, x\rangle$
$\sum_y |x, y\rangle$

Summary

- elfs process, random walk coupling

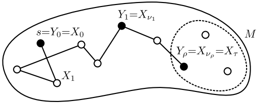


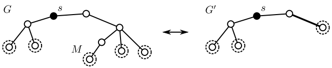


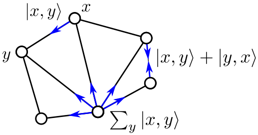


Summary

- elfs process, random walk coupling
- electric hitting time, $O(\log n)$ on trees

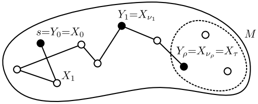


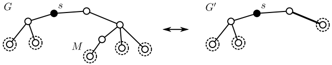


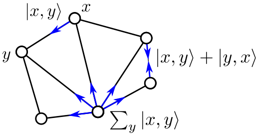


Summary

- elfs process, random walk coupling
- electric hitting time, $O(\log n)$ on trees
- quantum walk algorithms







Summary

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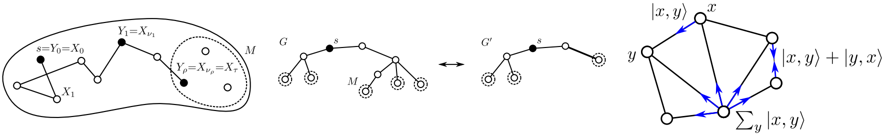
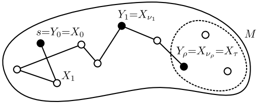
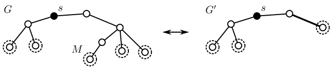


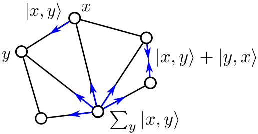
Open questions

Summary

- elfs process, random walk coupling
- electric hitting time, $O(\log n)$ on trees
- quantum walk algorithms







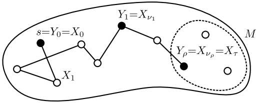
Open questions

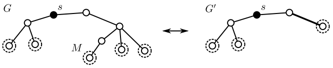
- electric flow sampling with random walks?

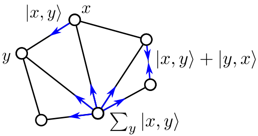
Elfs, trees and quantum walks

Summary

- elfs process, random walk coupling
- electric hitting time, $O(\log n)$ on trees
- quantum walk algorithms







Open questions

- electric flow sampling with random walks?
- explore quantum applications

Summary

- elfs process, random walk coupling
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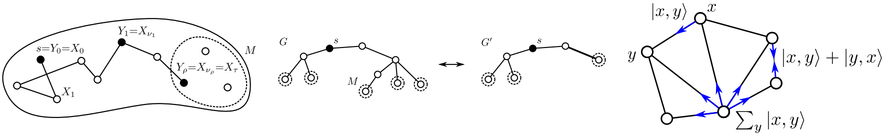
- quantum walk algorithms



Open questions

- electric flow sampling with random walks?
- explore quantum applications

Thank you!