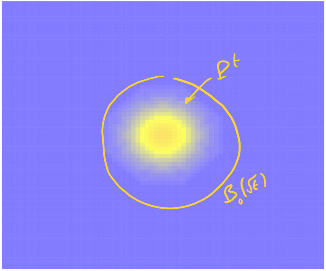
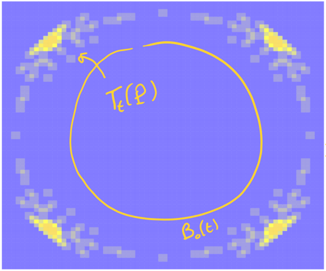
Quantum walks, the discrete wave equation and Chebyshev polynomials



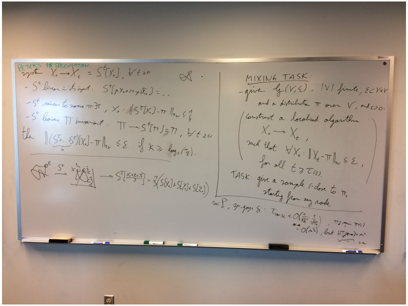


Simon Apers (CNRS & IRIF, Paris)
with Laurent Miclo

PCQT-IQC workshop, Waterloo (Canada), May '24

blast from the past

Calgary '17



blast from the past

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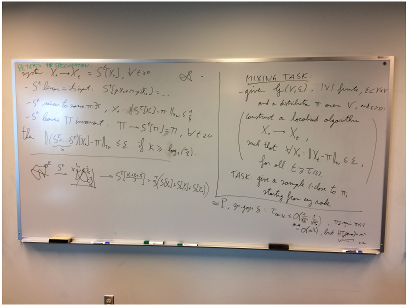

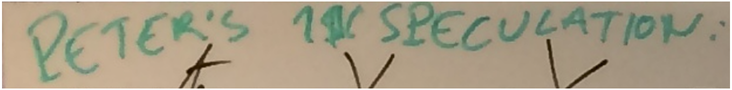

RWs from diffusion equation

QWs from wave equation

VC bound

converse VC bound?

RWs FROM DIFFUSION EQUATION

QWs FROM WAVE EQUATION

VC BOUND

CONVERSE VC BOUND?

Diffusion equation

$$\dot{u} = \Delta u$$

(over $\mathbb{R}^d$)

Diffusion equation

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$$\downarrow$$

$$u(t+1) = Pu(t)$$

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time $\mathbb{Z} \to \varepsilon\mathbb{Z}$, space $\mathbb{Z}^d \to \sqrt{\varepsilon}\mathbb{Z}^d$

$$u_\varepsilon(t+\varepsilon) = P_{\sqrt{\varepsilon}} u_\varepsilon(t)$$

then

$$\lim_{\varepsilon \to 0} u_\varepsilon(t_\varepsilon, x_\varepsilon) = u(t, x)$$

(for all t, x and $\lim_{\varepsilon\to 0} t_\varepsilon = t$, $\lim_{\varepsilon\to 0} x_\varepsilon = x$)

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generalizing:
RW over graph G

RWs from diffusion equation

QWs from wave equation

VC bound

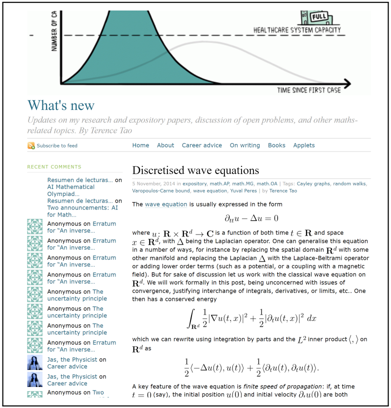
converse VC bound?

or,

spotting block encodings in the wild

or,

spotting block encodings in the wild



What's new

Updates on my research and expository papers, discussion of open problems, and other maths-related topics. By Terence Tao

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Discretised wave equations

5 November, 2014 in expository, math.AP, math.MG, math.OA | Tags: Cayley graphs, random walks, Varopoulos-Carne bound, wave equation, Yuval Peres | by Terence Tao

The wave equation is usually expressed in the form

$$\partial_{tt} u - \Delta u = 0$$

where $u: \mathbf{R} \times \mathbf{R}^d \to \mathbf{C}$ is a function of both time $t \in \mathbf{R}$ and space $x \in \mathbf{R}^d$, with Δ being the Laplacian operator. One can generalise this equation in a number of ways, for instance by replacing the spatial domain $\mathbf{R}^d$ with some other manifold and replacing the Laplacian Δ with the Laplace-Beltrami operator or adding lower order terms (such as a potential, or a coupling with a magnetic field). But for sake of discussion let us work with the classical wave equation on $\mathbf{R}^d$. We will work formally in this post, being unconcerned with issues of convergence, justifying interchange of integrals, derivatives, or limits, etc.. One then has a conserved energy

$$\int_{\mathbf{R}^d} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}|\partial_t u(t,x)|^2\ dx$$

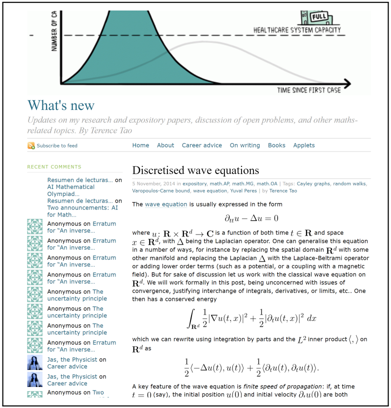
which we can rewrite using integration by parts and the L^2 inner product $\langle , \rangle$ on $\mathbf{R}^d$ as

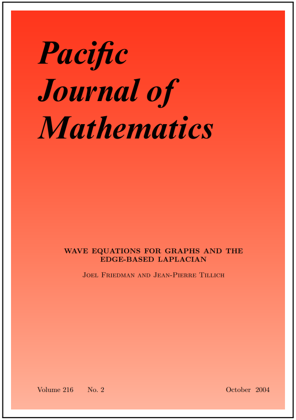
$$\frac{1}{2}\langle -\Delta u(t), u(t)\rangle + \frac{1}{2}\langle \partial_t u(t), \partial_t u(t)\rangle.$$

A key feature of the wave equation is *finite speed of propagation*: if, at time $t = 0$ (say), the initial position $u(0)$ and initial velocity $\partial_t u(0)$ are both

or,

spotting block encodings in the wild





Wave equation

$$\dot{u} = v,$$
$$\dot{v} = \Delta u$$

Wave equation

$$
\begin{gathered}
\dot{u} = v, \\
\dot{v} = \Delta u \\
\downarrow \\
u(t+1) = Pu(t) + v(t), \\
v(t+1) = -(I - P^2)u(t) + Pv(t)
\end{gathered}
$$

Wave equation

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$$v_\varepsilon(t+\varepsilon) = -\frac{1}{\varepsilon}(I - P_\varepsilon^2)u_\varepsilon(t) + P_\varepsilon v_\varepsilon(t)$$

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! (energy) conservation

$$E = \|\sqrt{I - P^2}u\|^2 + \|v\|^2$$

Wave equation

change of variables: $w = \sqrt{I - P^2}u$

$$\begin{bmatrix} w(t+1) \\ v(t+1) \end{bmatrix} = \begin{bmatrix} P & \sqrt{I - P^2} \\ -\sqrt{I - P^2} & P \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}$$

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t steps:

$$\begin{bmatrix} P & \sqrt{I-P^2} \\ -\sqrt{I-P^2} & P \end{bmatrix}^t = \begin{bmatrix} T_t(P) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

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P RW over graph G

$\downarrow$

discrete wave equation or quantum walk over G

RWs FROM DIFFUSION EQUATION

QWs FROM WAVE EQUATION

VC BOUND

CONVERSE VC BOUND?

random walks and Chebyshev polynomials?

random walks and Chebyshev polynomials?

varopoulos-carne

Highlight all   Match case   Whole words

1 of 14 matches

(from “Probability on Trees and Networks” by Lyons and Peres)

random walks and Chebyshev polynomials?

varopoulos-carne

Highlight all Match case Whole words

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Varopoulos-Carne bound:

$$P^t(x,y) \le e^{-d(x,y)^2/t}$$

Varopoulos-Carne bound

Chebyshev expansion

$$P^t = \sum_{k=0}^{t} \alpha_k T_k(P)$$

Varopoulos-Carne bound

Chebyshev expansion

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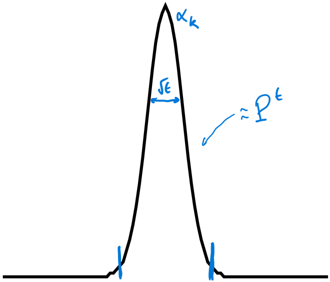

Varopoulos-Carne bound

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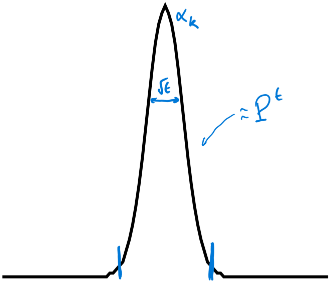

$$\approx_\varepsilon \sum_{k=0}^{\sqrt{t \log(1/\varepsilon)}} \alpha_k T_k(P) \qquad \square$$

Varopoulos-Carne bound

corollary 1: mixing time $\tau \geq \mathrm{diam}(G)^2 / \log(n)$

Varopoulos-Carne bound

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corollary 2: block encoding $\begin{bmatrix} P^t & \cdot \\ \cdot & \cdot \end{bmatrix}$ using $\sqrt{t}$ QW steps

QUANTUM FAST-FORWARDING:
MARKOV CHAINS AND GRAPH PROPERTY TESTING

'19

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{*simon.apers@inria.fr, alain.sarlette@inria.fr*}

Varopoulos-Carne bound

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basis for QW search in $\sqrt{\text{hitting time}}$

Quadratic speedup for finding marked vertices by Quantum walks

Authors: Andris Ambainis, András Gilyén, Stacey Jeffery, Martins Kokainis Authors Info & Claims

STOC 2020: Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing • June 2020 • Pages 412–424
• https://doi.org/10.1145/3357713.3384252

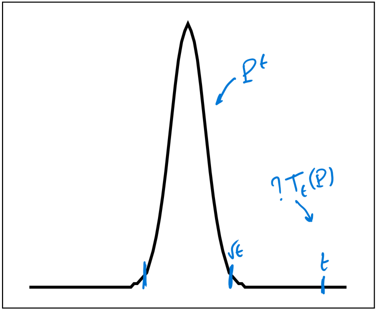
RWs from diffusion equation

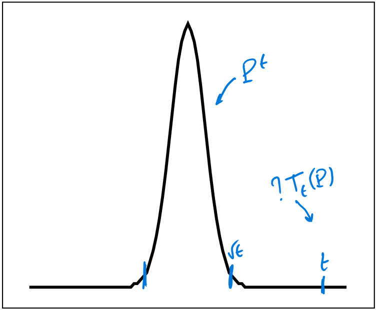
QWs from wave equation

VC bound

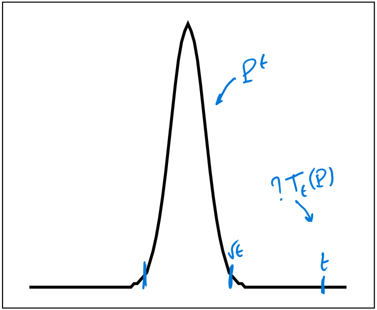
converse VC bound?


DIFFUSION WAVE-MOTION
[SPELL CARD]


P^t
? $T_t(P)$
$\sqrt{t}$
t



VC bound: random walks diffusive



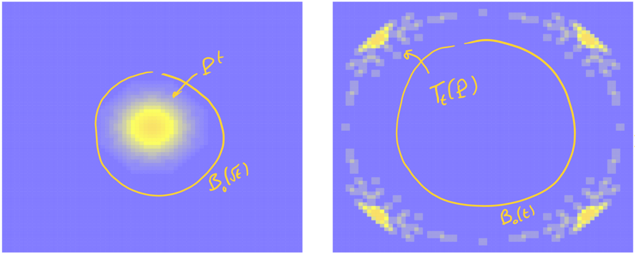
VC bound: random walks diffusive

converse VC bound: quantum walks ballistic?

$$\sum_{y \notin B_x(t/2)} T_t(P)^2_{x,y} \in \Omega(1)$$

Converse VC bound?

yes (and known) for lattices



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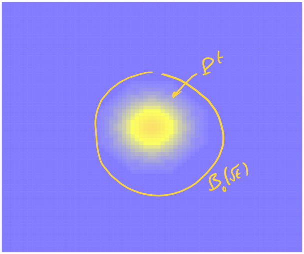

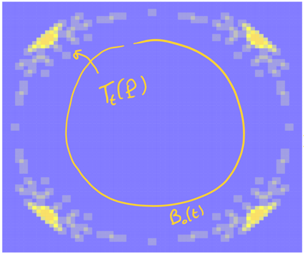

A-Miclo (arXiv:2402.07809):

weak limit for quantum walks on lattices

Converse VC bound?

RW on $\mathbb{Z}$:

$$P = \frac{1}{2}P_{\leftarrow} + \frac{1}{2}P_{\rightarrow}$$

Converse VC bound?

RW on $\mathbb{Z}$:

$$P = \frac{1}{2}P_{\leftarrow} + \frac{1}{2}P_{\rightarrow}$$

$$T_t(P) = \frac{1}{2}\left(P_{\leftarrow}\right)^t + \frac{1}{2}\left(P_{\rightarrow}\right)^t$$

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RW on $\mathbb{Z}^2$:

$$P = \frac{1}{2}P_{\leftrightarrow} + \frac{1}{2}P_{\updownarrow}$$

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Converse VC bound?

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! commuting variables $x = P_{\leftrightarrow}$ and $y = P_{\updownarrow}$ so

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Converse VC bound?

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$\rightarrow$ weak limit induced by $\sum a_{p,q}^2 \delta_{p/t,q/t}$

Converse VC bound?

more generally (open questions):

quantum walk $T_t(P)$ on graphs?

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conjecture:
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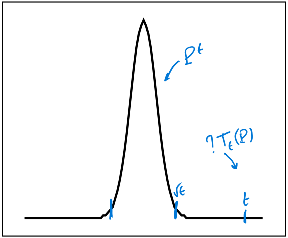
thanks!



Extra slides: discrete approximation

$$\boxed{\dot{u} = \Delta u}$$

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Extra slides: Gram-Schmidt walk

Gram-Schmidt on (ordered) set of vectors

$$\{\left|x\right\rangle, P\left|x\right\rangle, P^2\left|x\right\rangle, \ldots, P^t\left|x\right\rangle\}$$