Bounding the Convergence of Mixing and Consensus Algorithms

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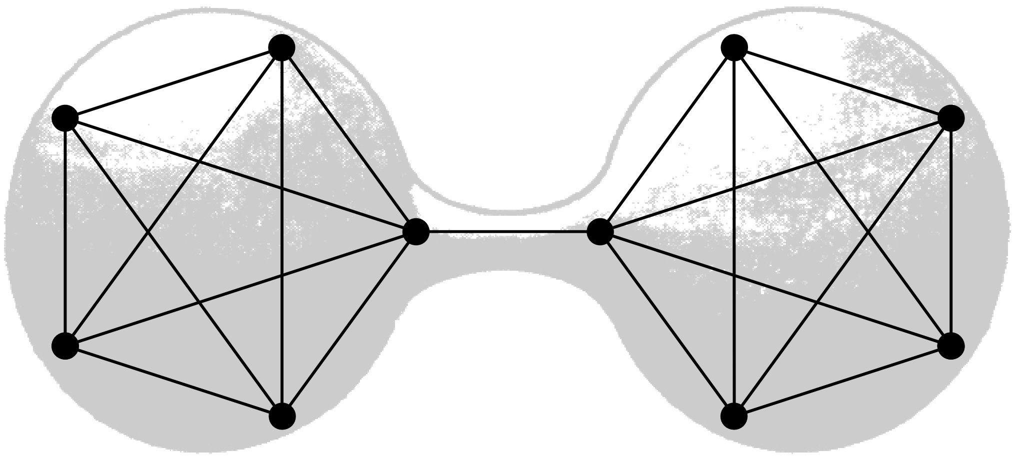
dynamics on graphs:

- diffusion
- rumour spreading
- weight balancing
- quantum walks
- ...



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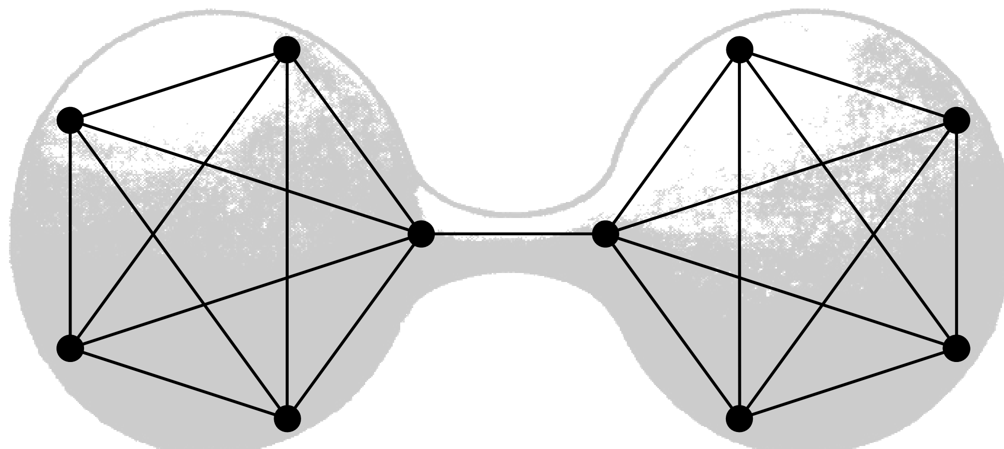

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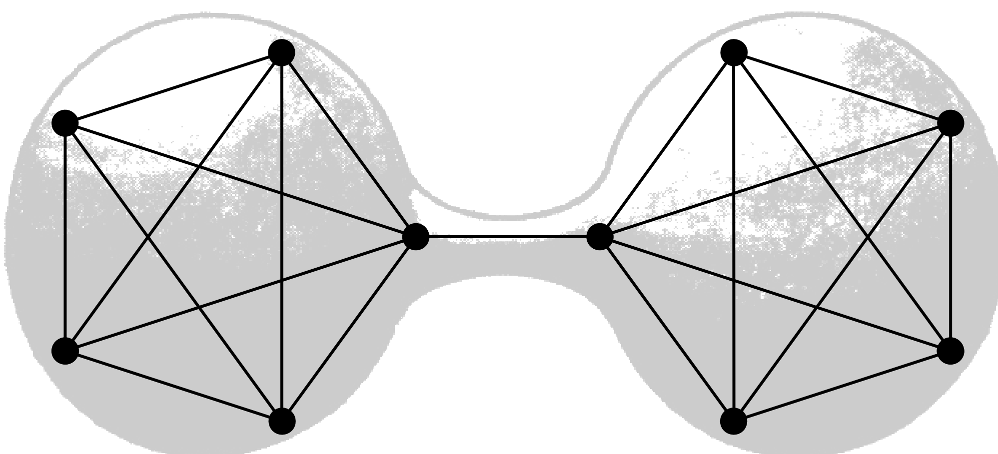

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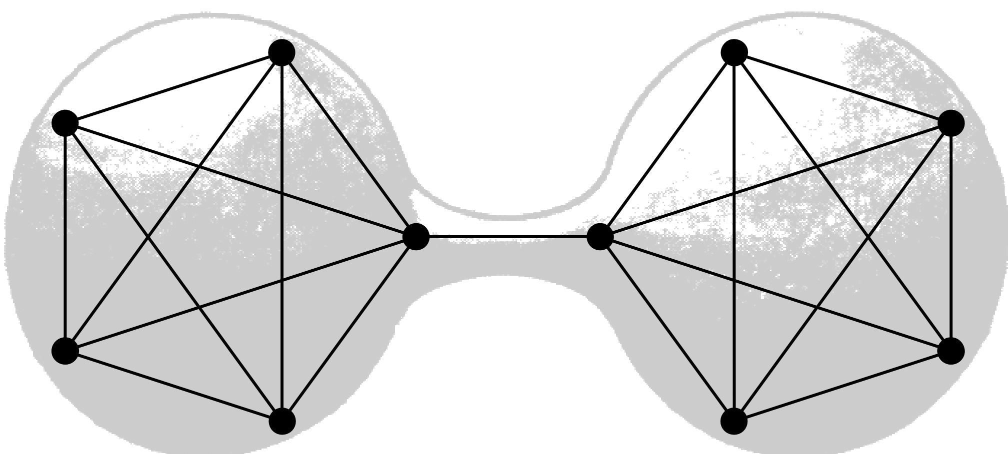
time scale = “mixing time”

example: random walk on dumbbell graph



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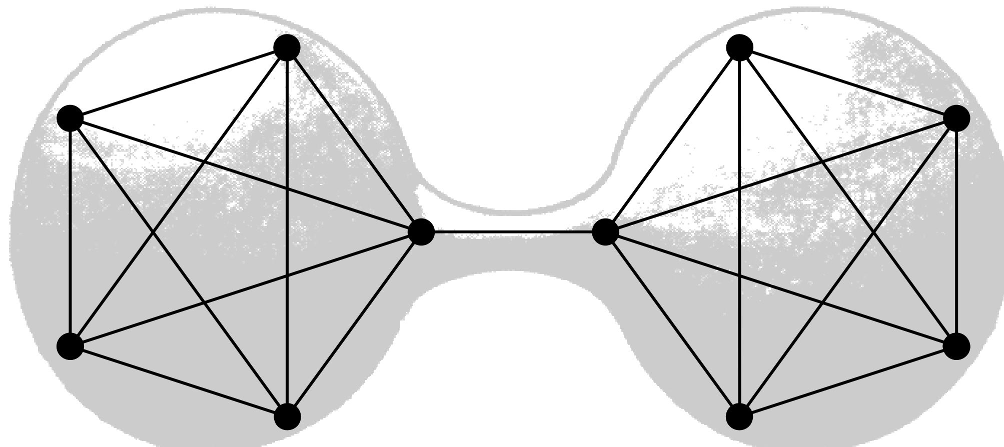
$$v_0 \xrightarrow{t} v_t = P^t v_0$$



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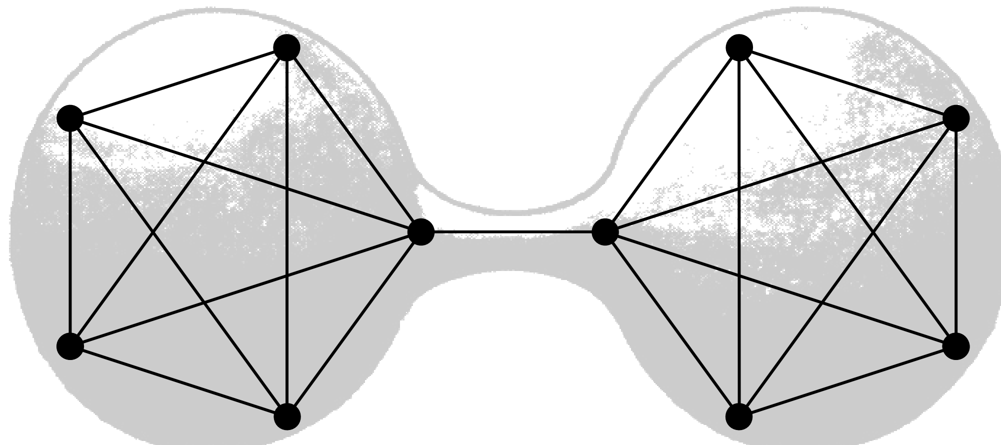

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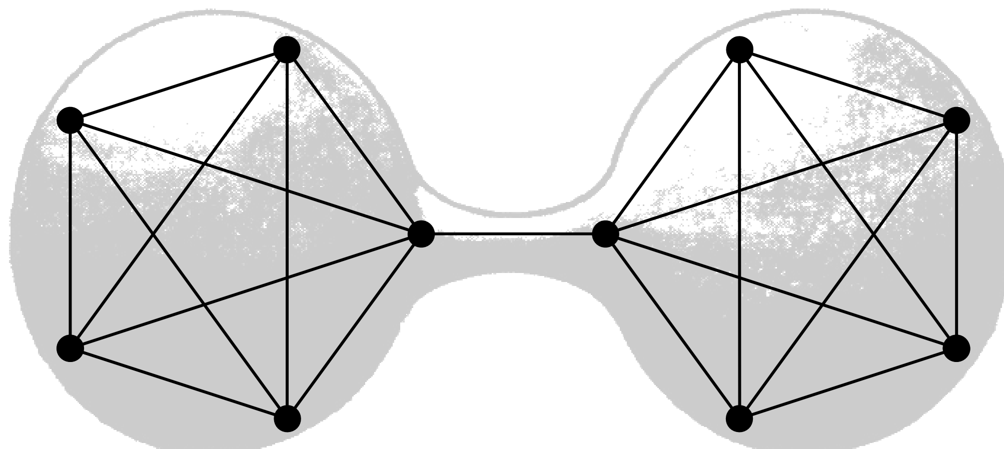
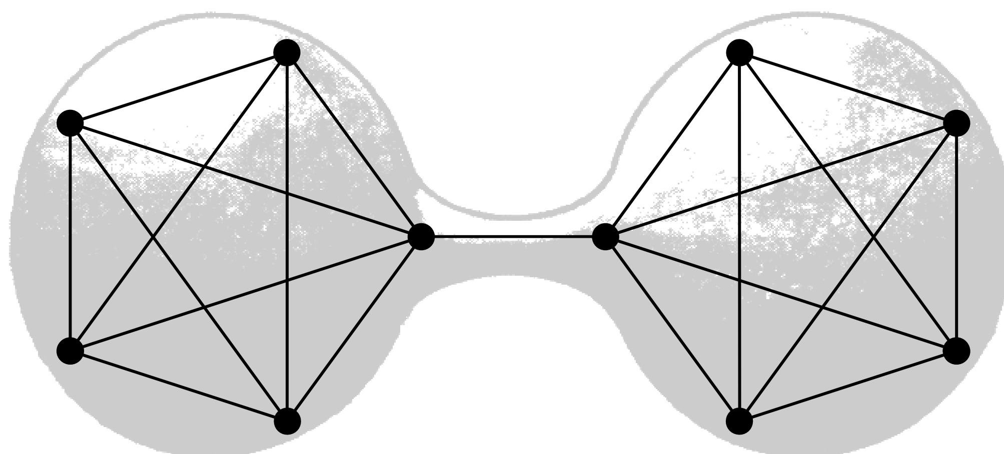


mixing time:

$$\tau \equiv \min \left\{ T \mid \|P^t v_0 - \pi\|_1 \leq \tfrac{1}{2}, \forall t \geq T, v_0 \right\}$$

example: random walk on dumbbell graph

$$\tau \geq N^2$$



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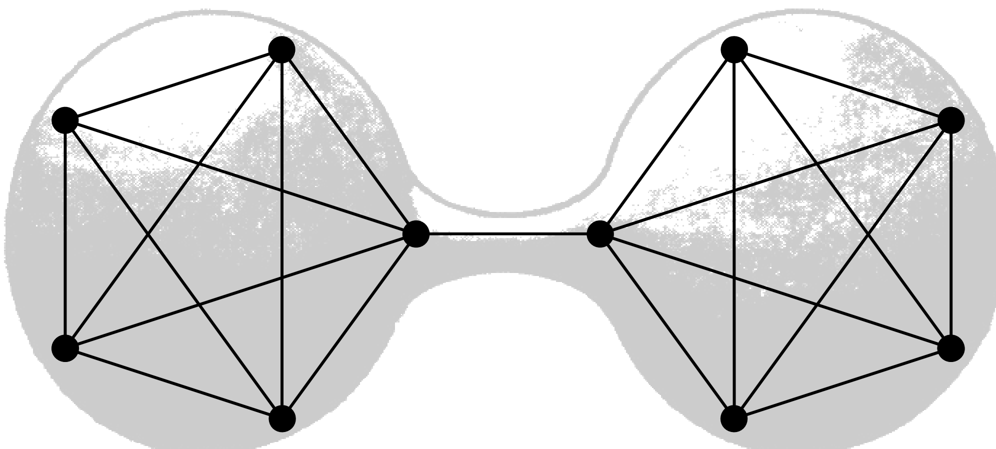

conductance bound: $\tau \geq \dfrac{1}{\Phi}$

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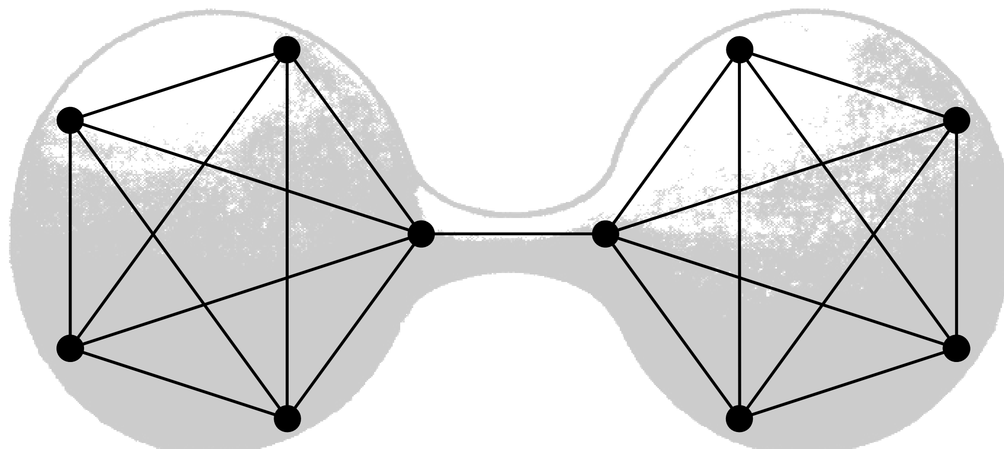

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$$\Phi = \min_{\pi(X) \leq \frac{1}{2}} \Phi(X) \qquad \Phi(X) = (P\pi_X)(X^c) = \frac{|E(X, X^c)|}{|E(X)|}$$

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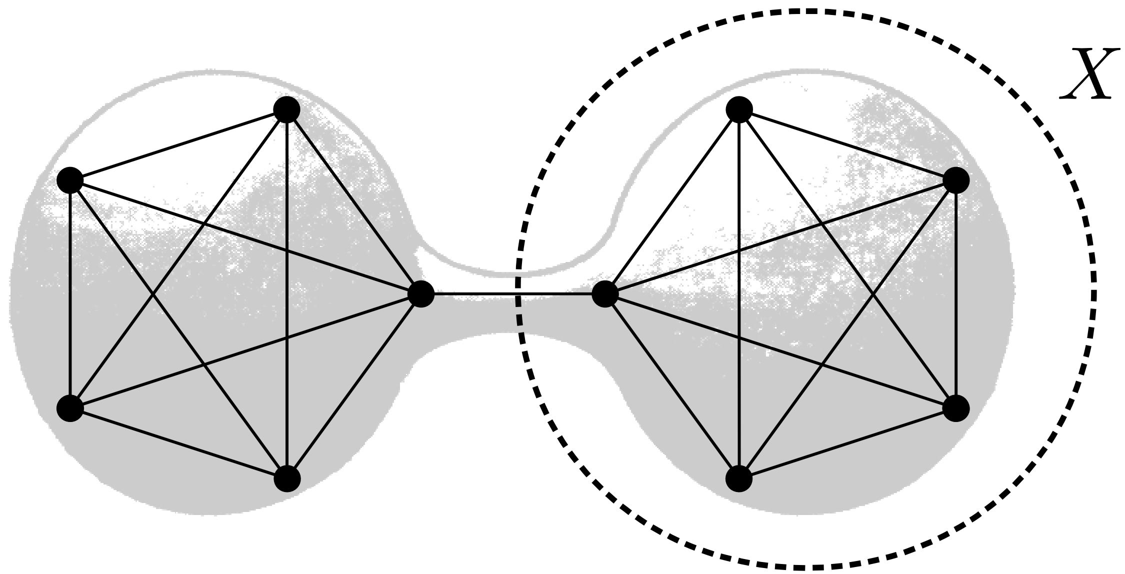

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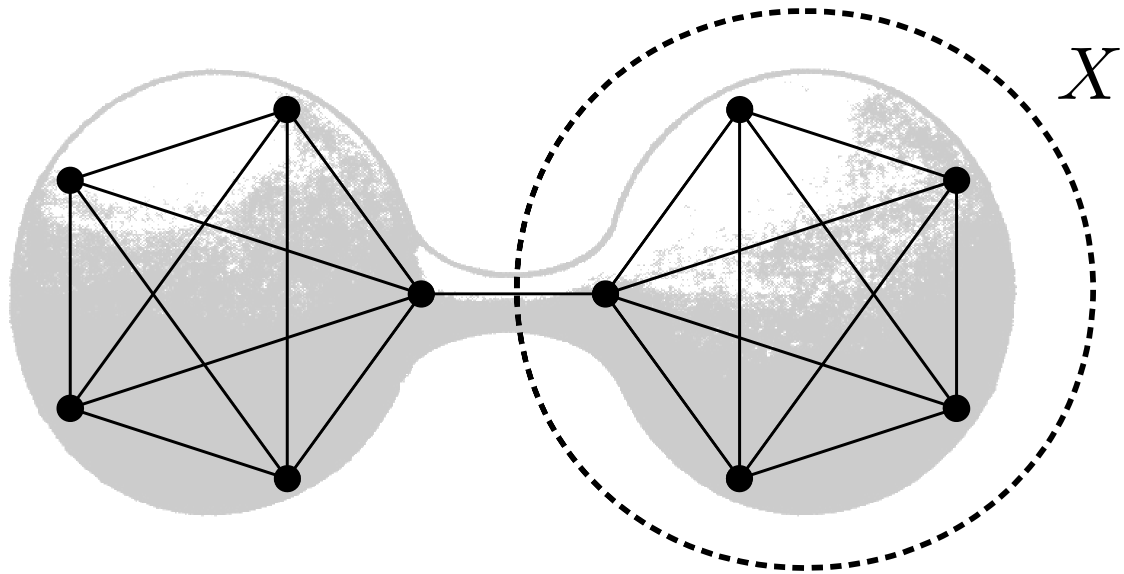

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proof idea:

$$(P^t \pi_X)(X^c) \leq t(P\pi_X)(X^c) = t\Phi(X)$$

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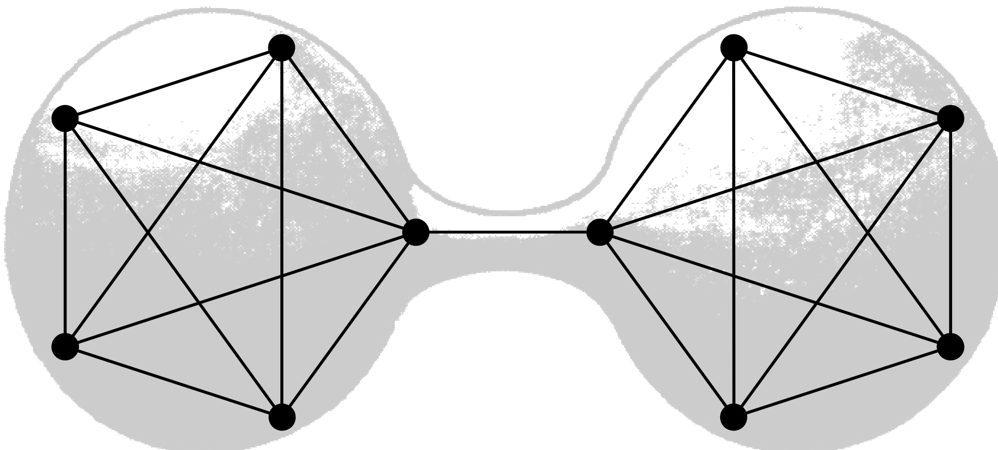

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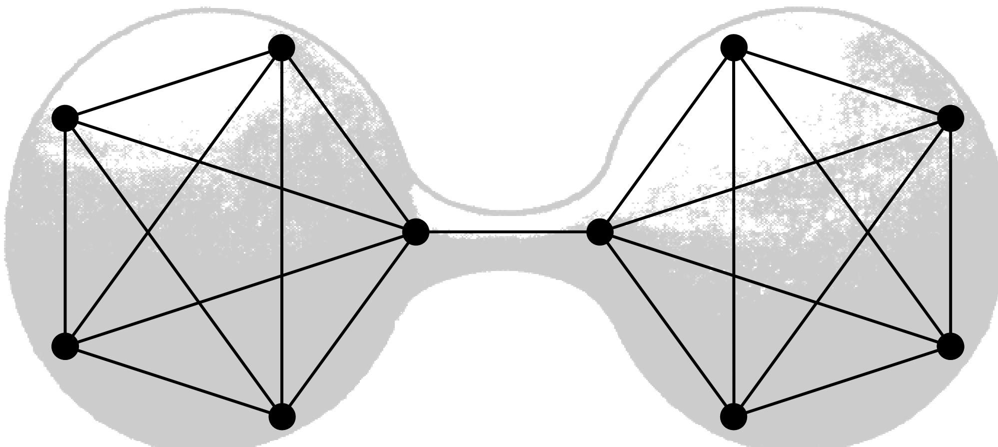
however, diameter = 3
can we do any better ?



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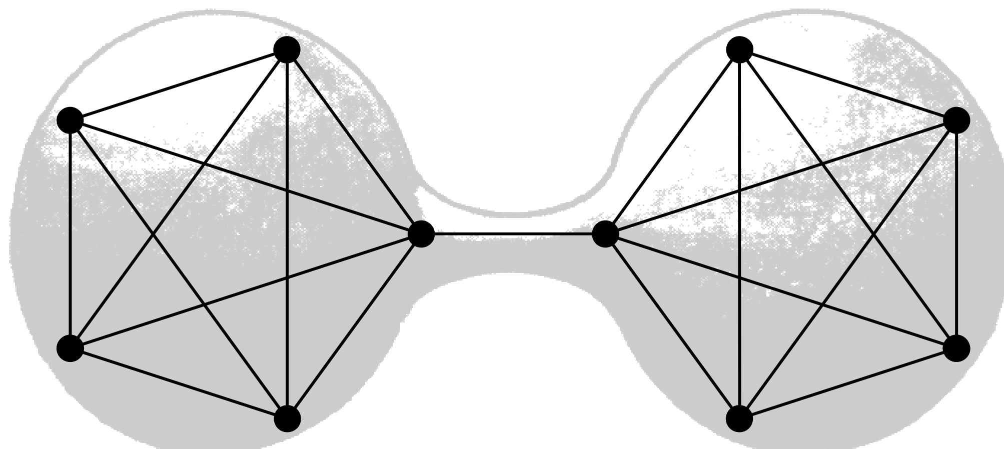

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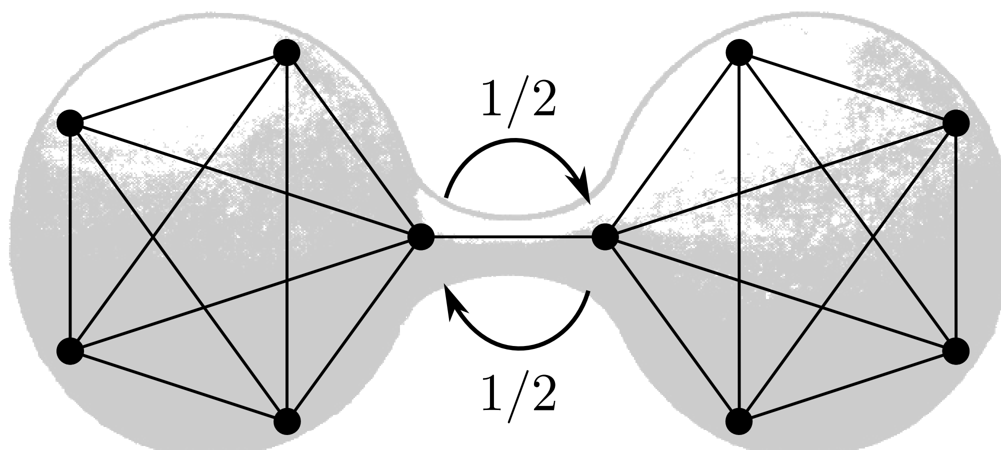

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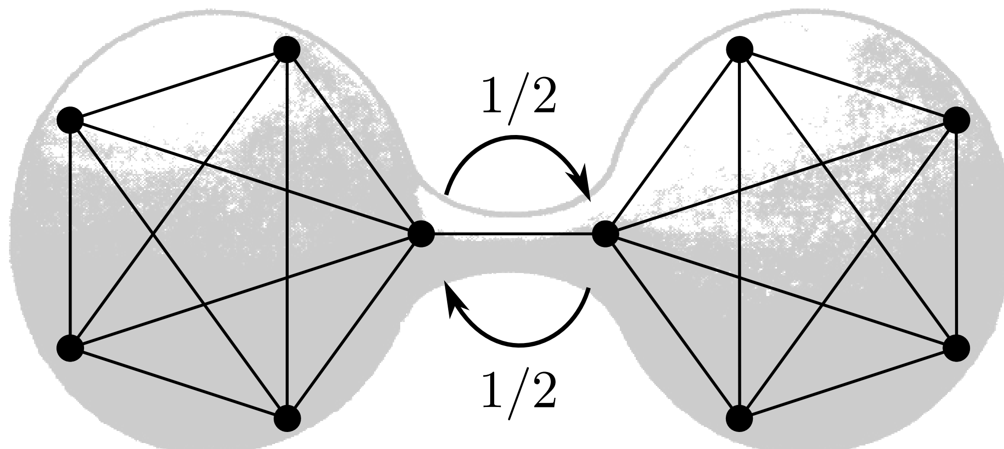

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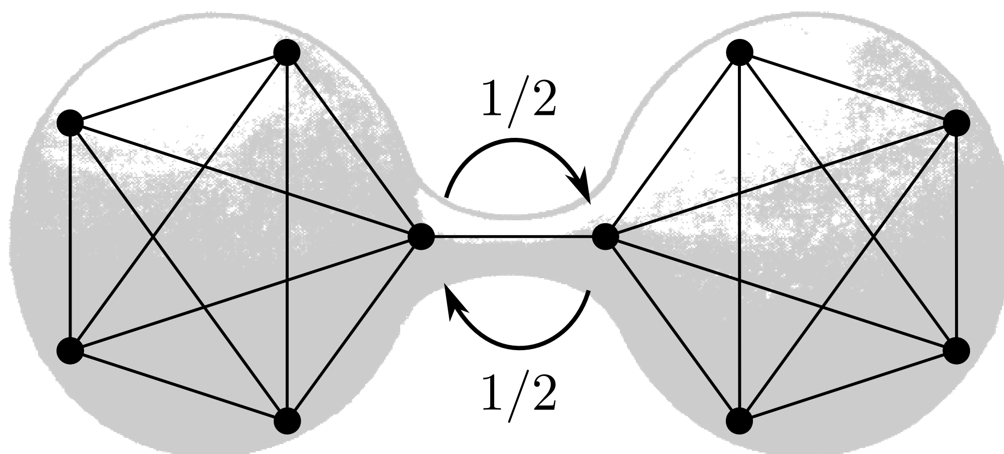

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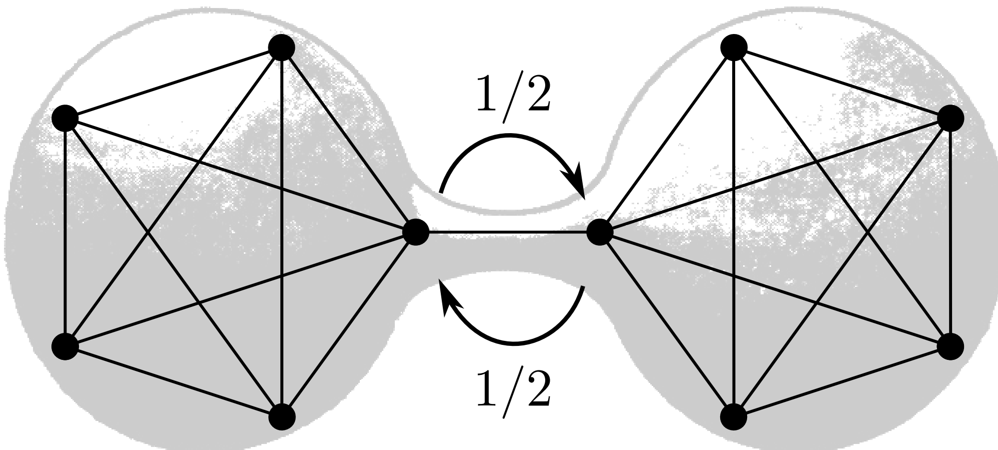

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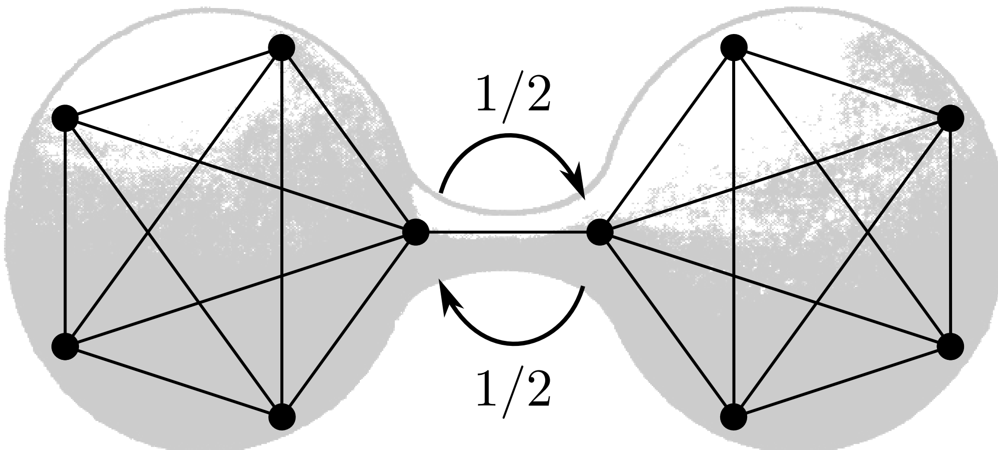

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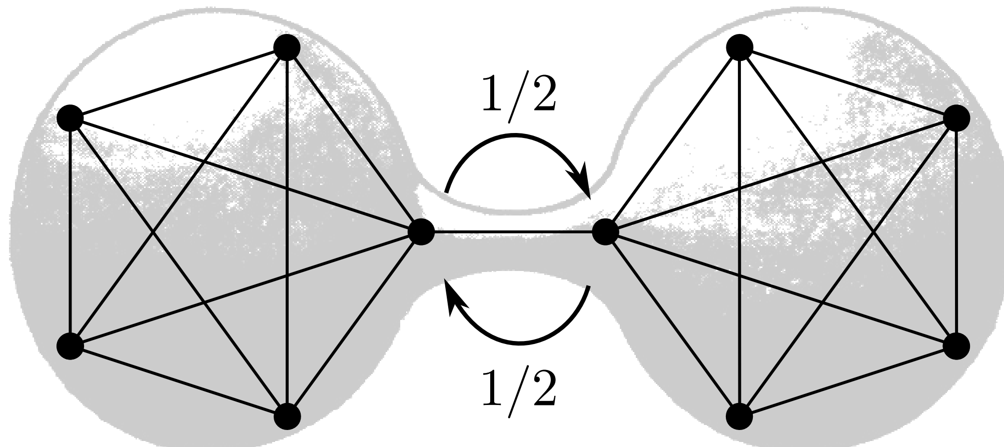

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not using simple Markov chains:

$$\Phi_{G,\pi} = \max_{P \sim G : P\pi = \pi} \Phi(P) \leq \frac{1}{N}$$

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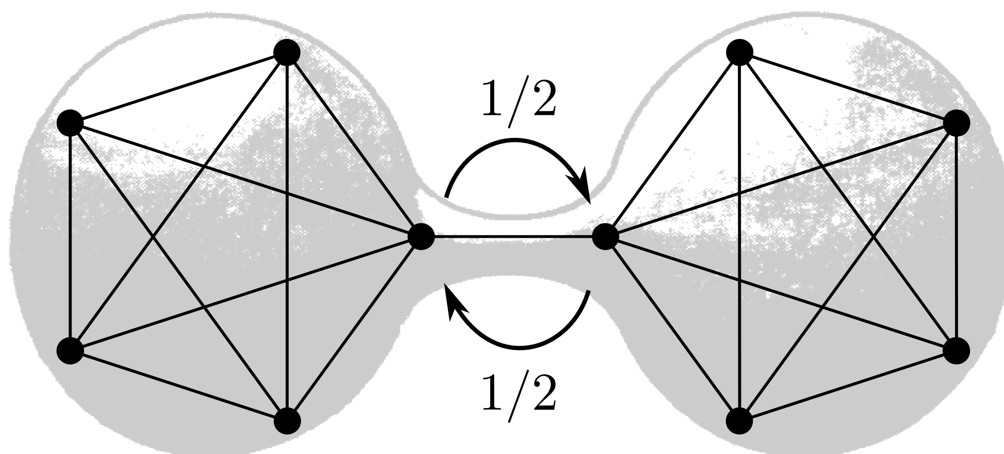

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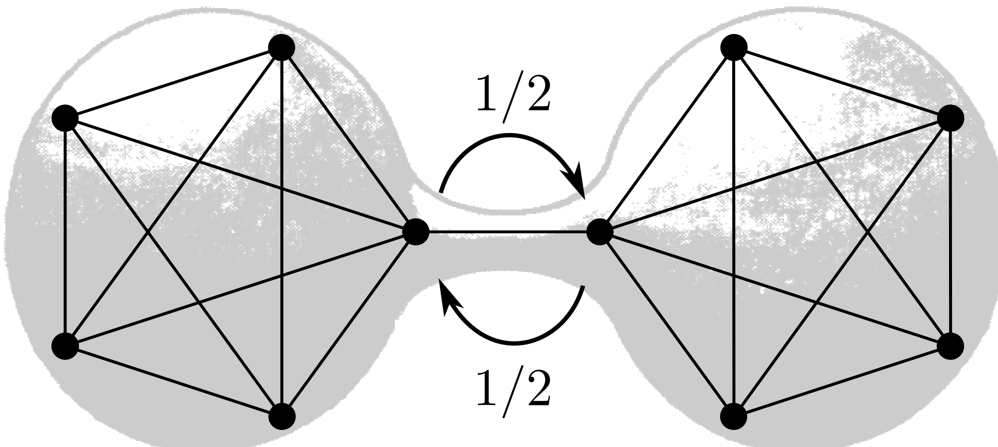

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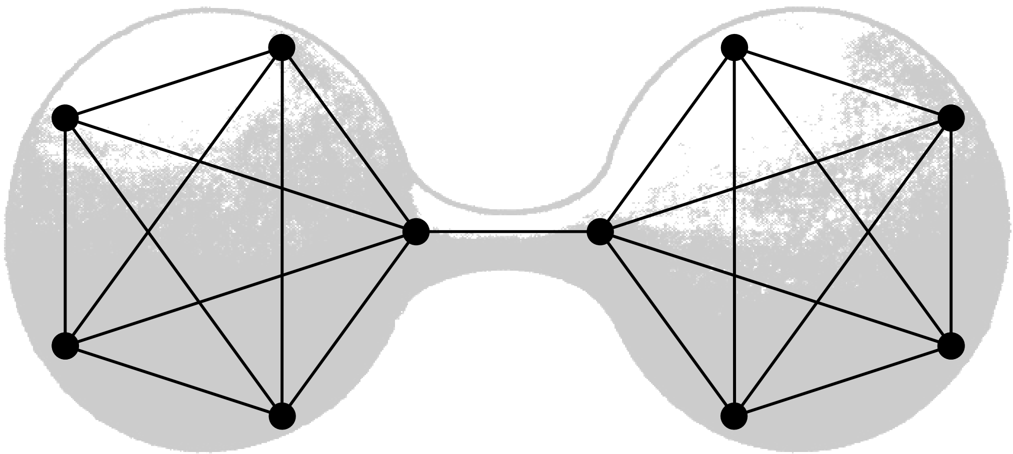
what if we allow time dependence? memory? quantum dynamics?

e.g. non-backtracking random walks, lifted Markov chains, simulated annealing, polynomial filters, quantum walks,...

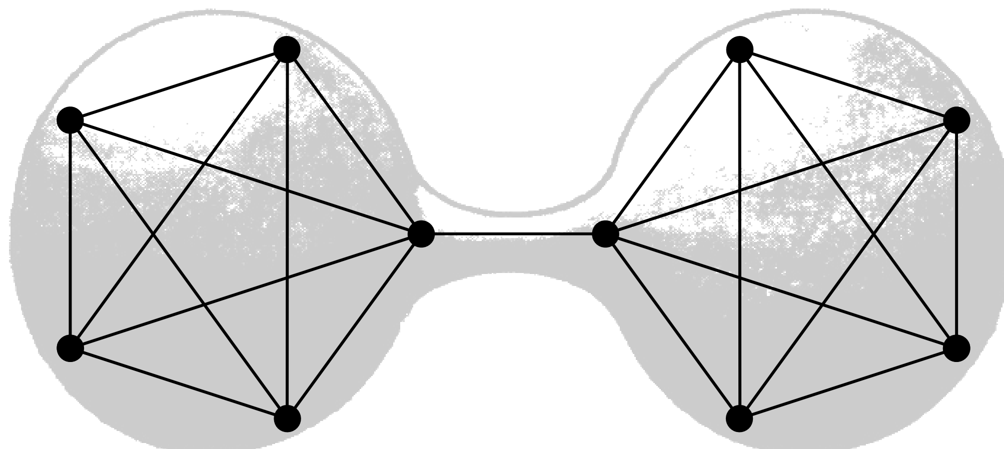
stochastic process



stochastic process

$$v_0 \xrightarrow{t} v_t$$

$$v_0 \xrightarrow{\infty} \pi$$



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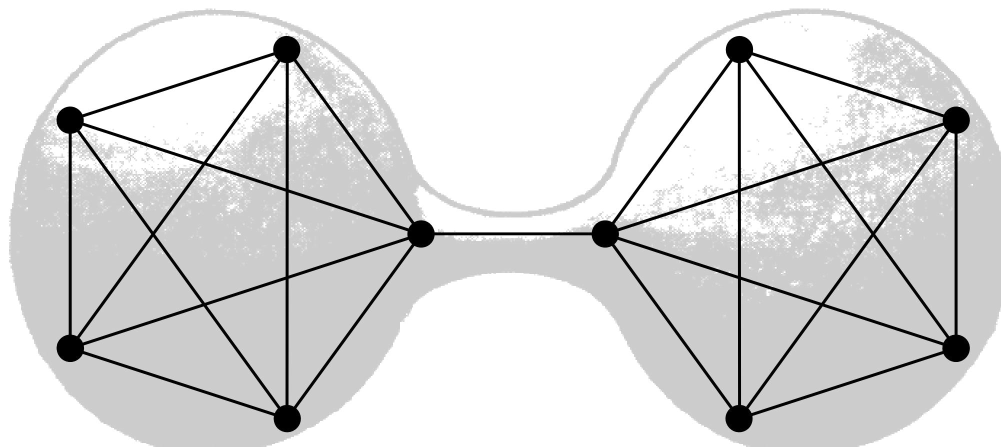

- linear $$p\,v_0 + (1-p)v_0' \xrightarrow{t} p\,v_t + (1-p)v_t'$$

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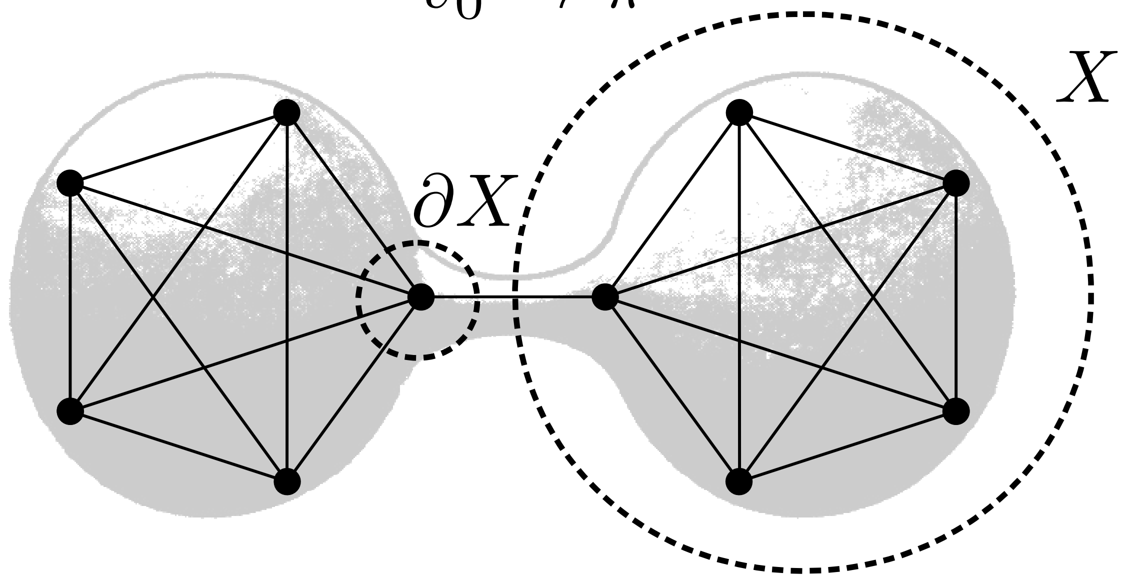

- linear $$p\, v_0 + (1-p)v_0' \xrightarrow{t} p\, v_t + (1-p)v_t'$$

- local $$v_{t+1}(X) \leq v_t(X) + v_t(\partial X)$$

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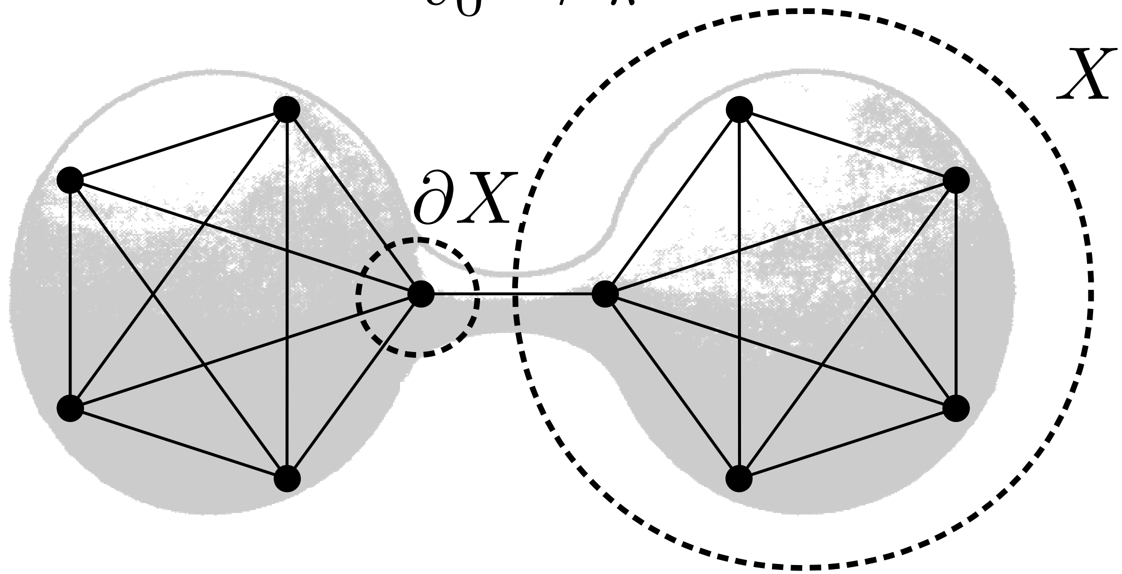

- linear $$p\, v_0 + (1-p) v'_0 \xrightarrow{t} p\, v_t + (1-p) v'_t$$
- local $$v_{t+1}(X) \leq v_t(X) + v_t(\partial X)$$
- invariant $$\pi \xrightarrow{t} \pi$$

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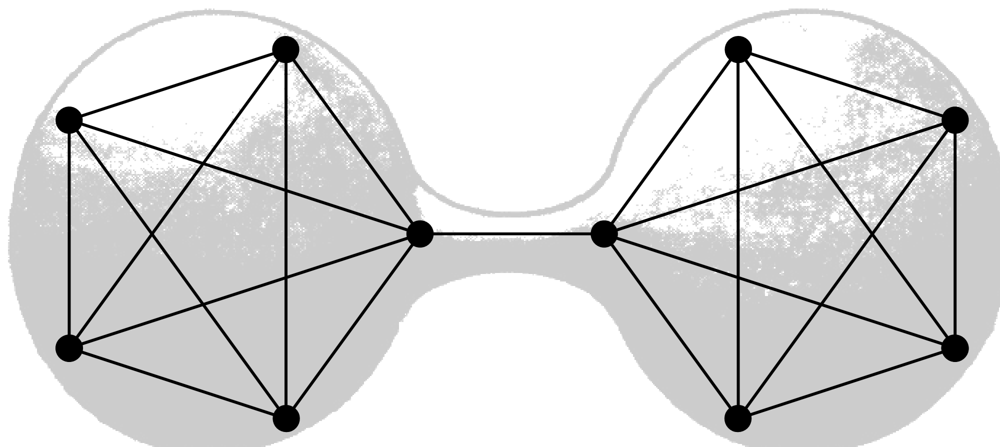

examples of linear, local and invariant stochastic processes:

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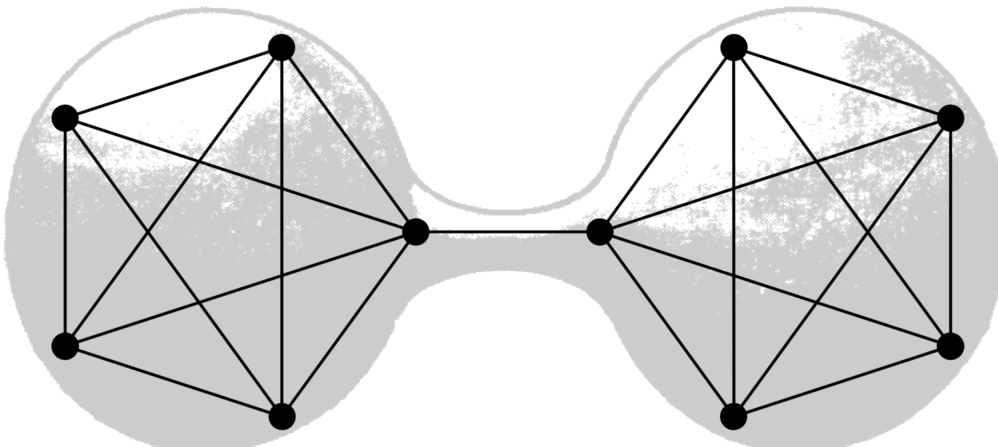

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- Markov chains, time-averaged MCs, time-inhomogeneous invariant MCs

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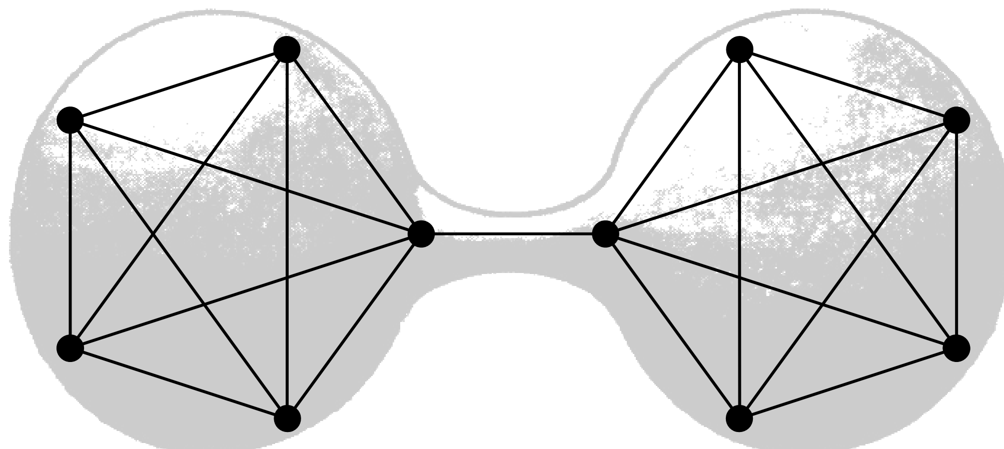

examples of linear, local and invariant stochastic processes:

- Markov chains, time-averaged MCs, time-inhomogeneous invariant MCs
- lifted MCs, non-backtracking RWs on regular graphs

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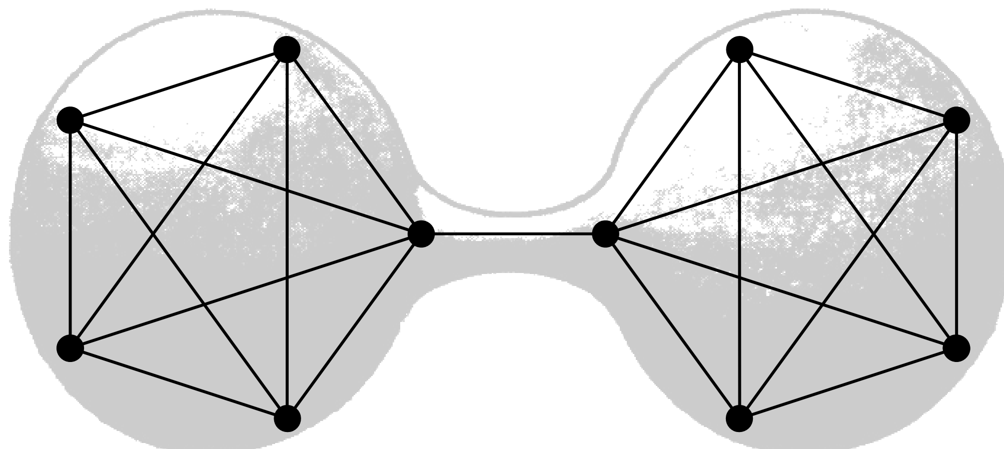

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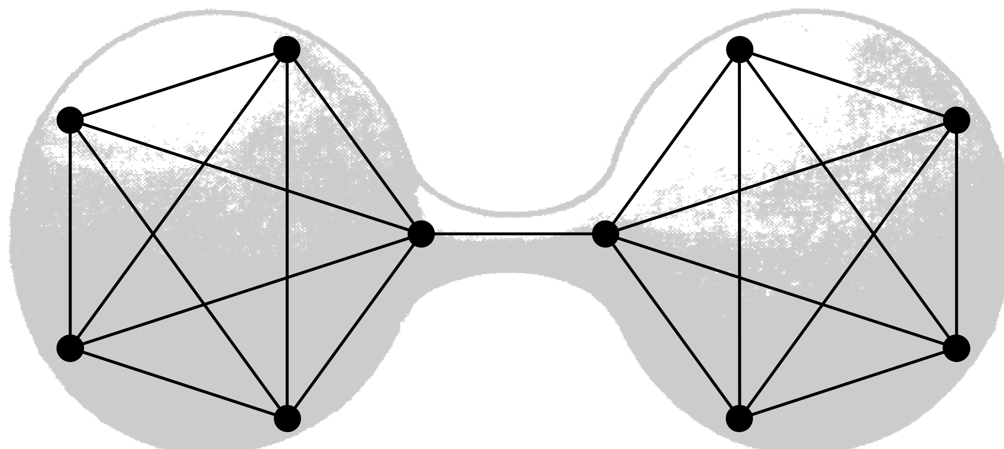

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- quantum walks and quantum Markov chains

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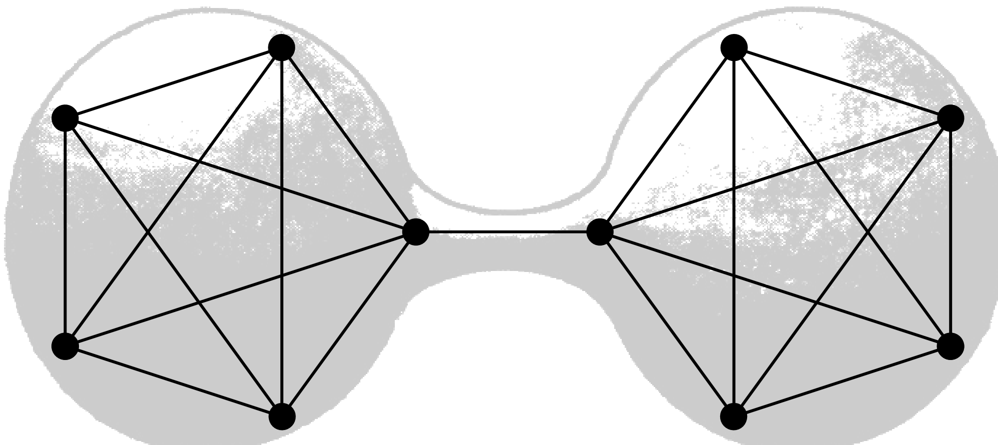

<u>main theorem:</u>

any linear, local and invariant stochastic process has a mixing time

$$\tau \geq \frac{1}{\Phi_{G,\pi}}$$

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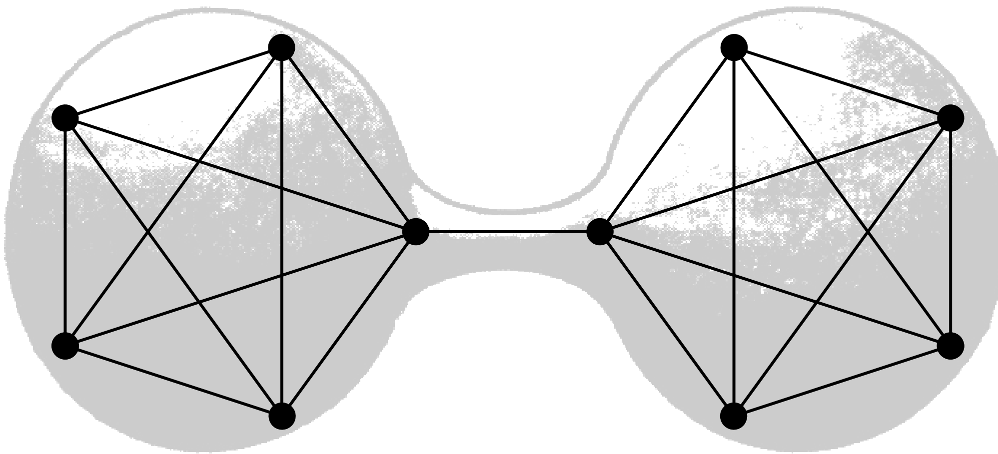

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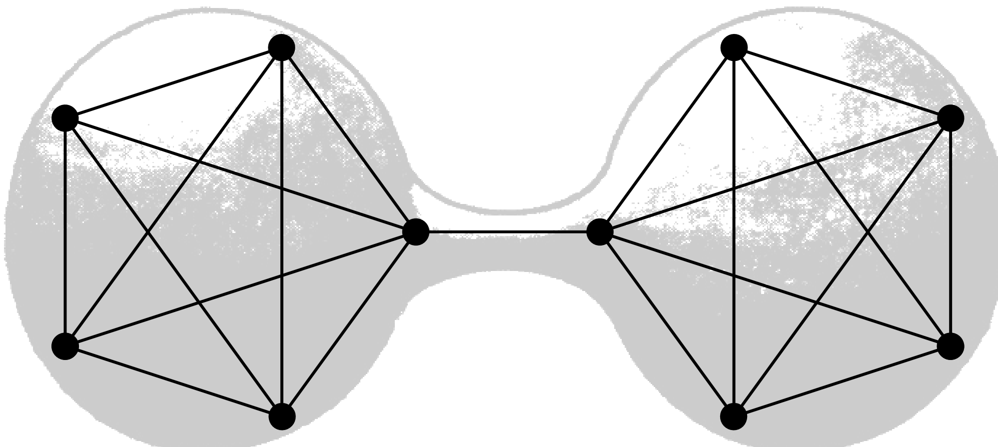
on dumbell graph:

$$\Phi_{G,\pi} = 1/N$$



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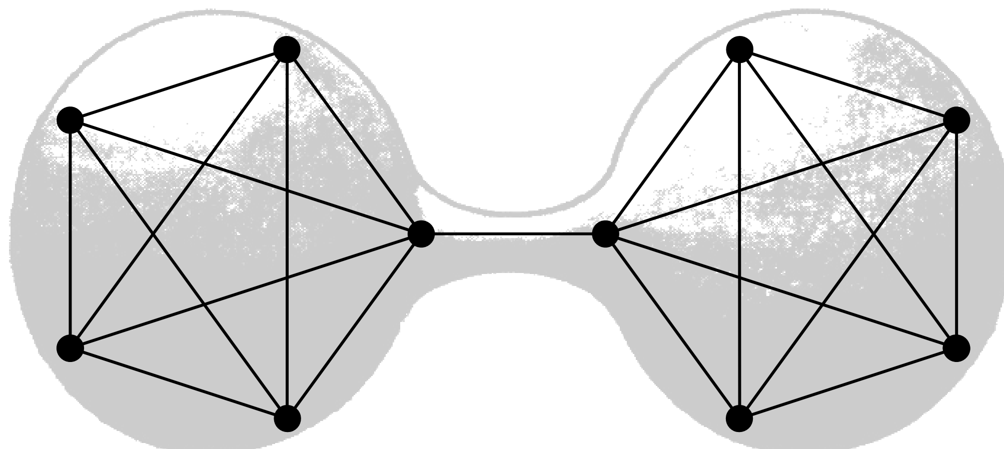

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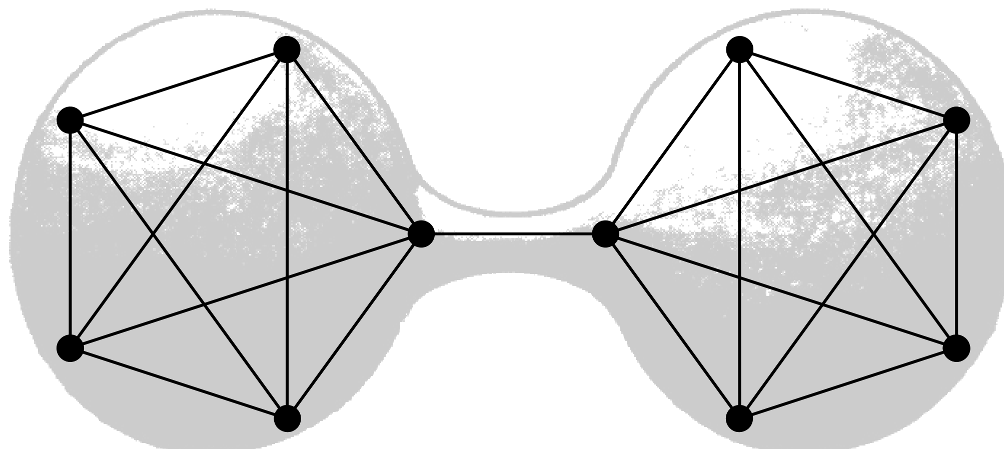

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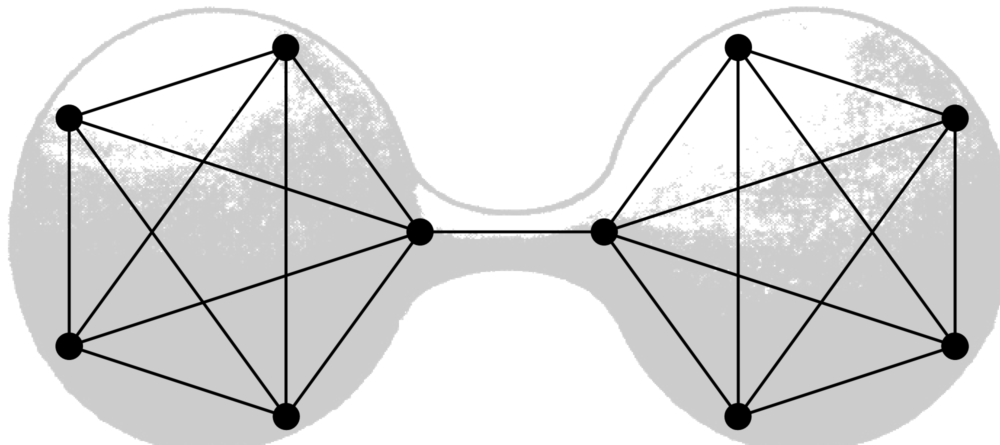

proof:

1) we build a Markov chain simulator

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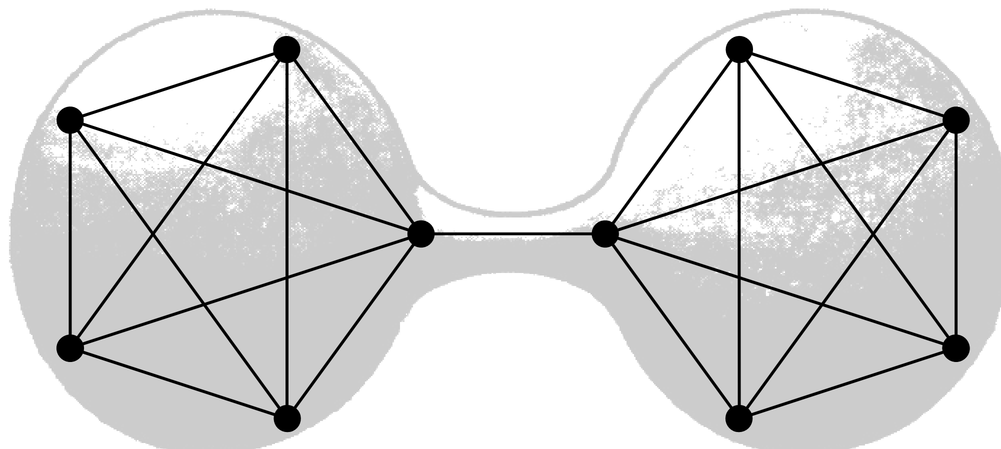

proof:

1) we build a Markov chain simulator

2) we prove the theorem for Markov chain simulator

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Stochastic bridge lemma: Let $v_0 \xrightarrow{t} v_t$ be a local stochastic process. For all $v_0, t \geq 0$, we can construct a local stochastic transition matrix $P_{t+1}^{(v_0)} \sim G$ such that

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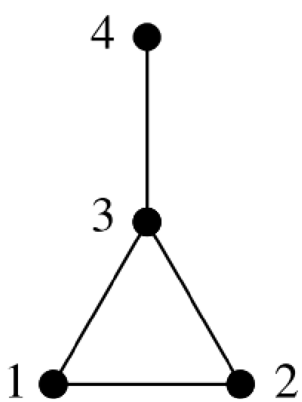

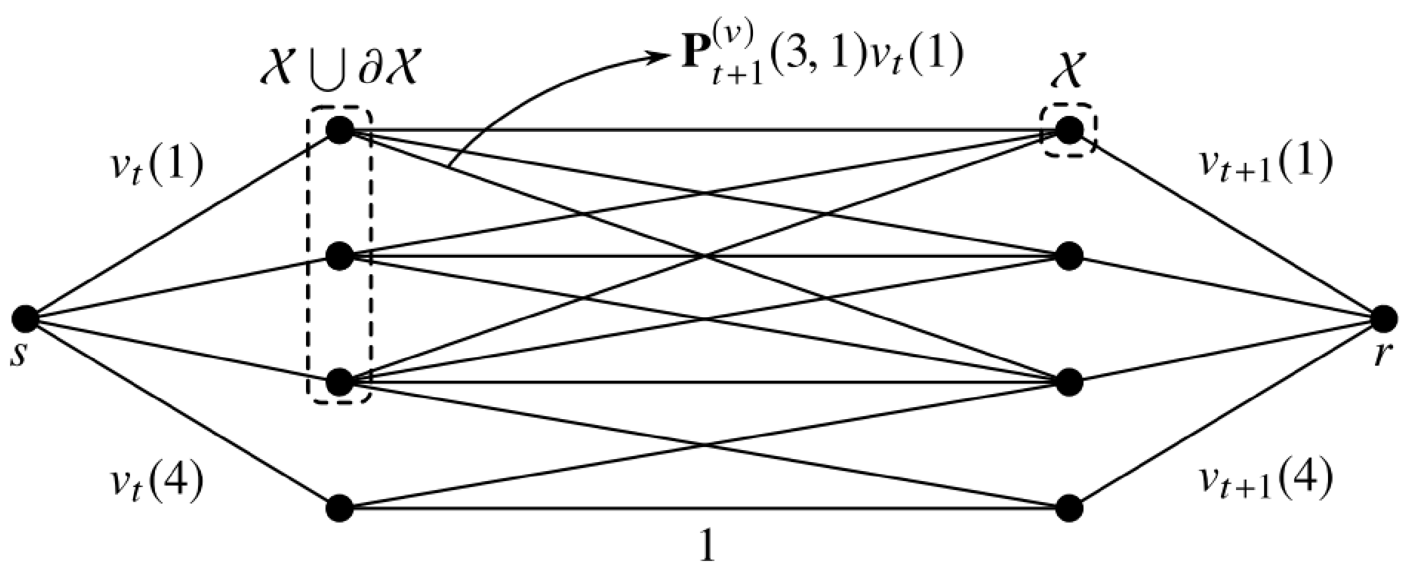

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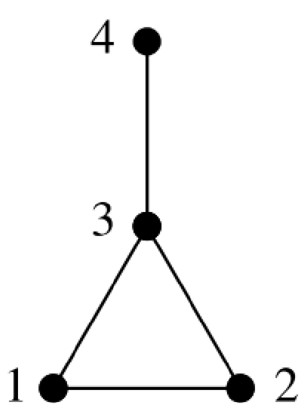

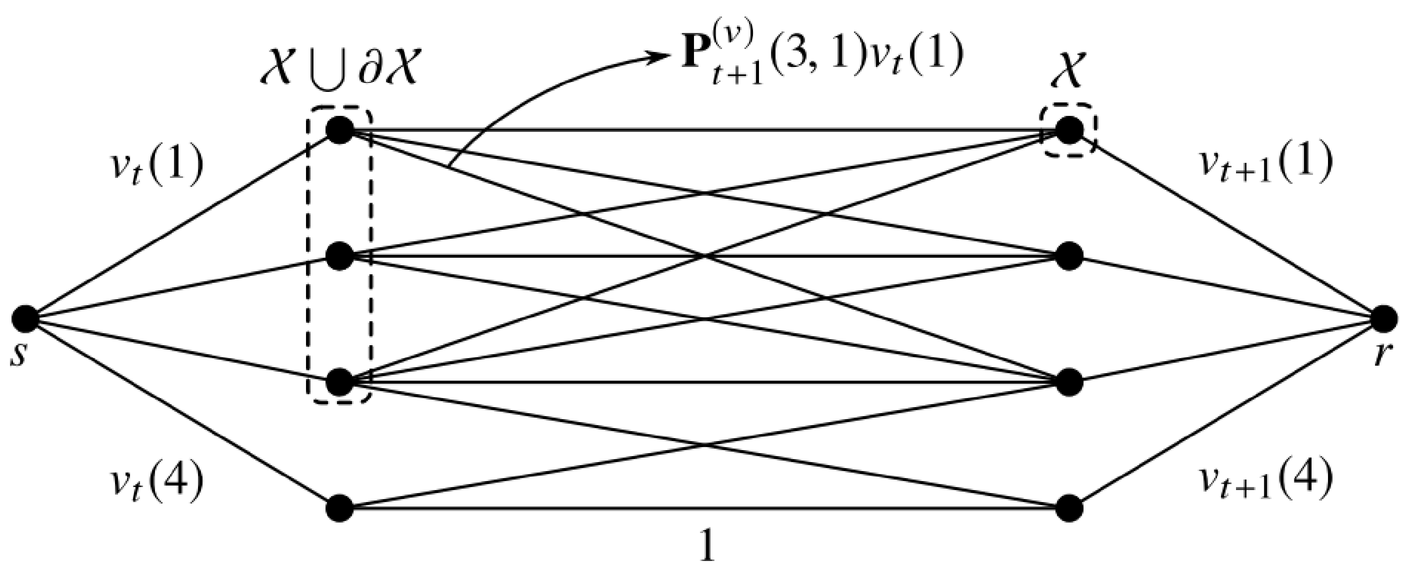

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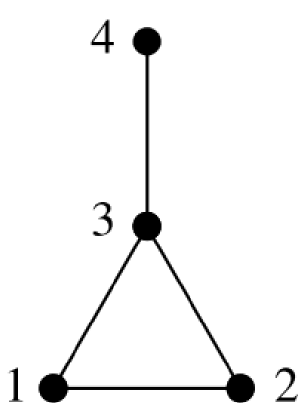
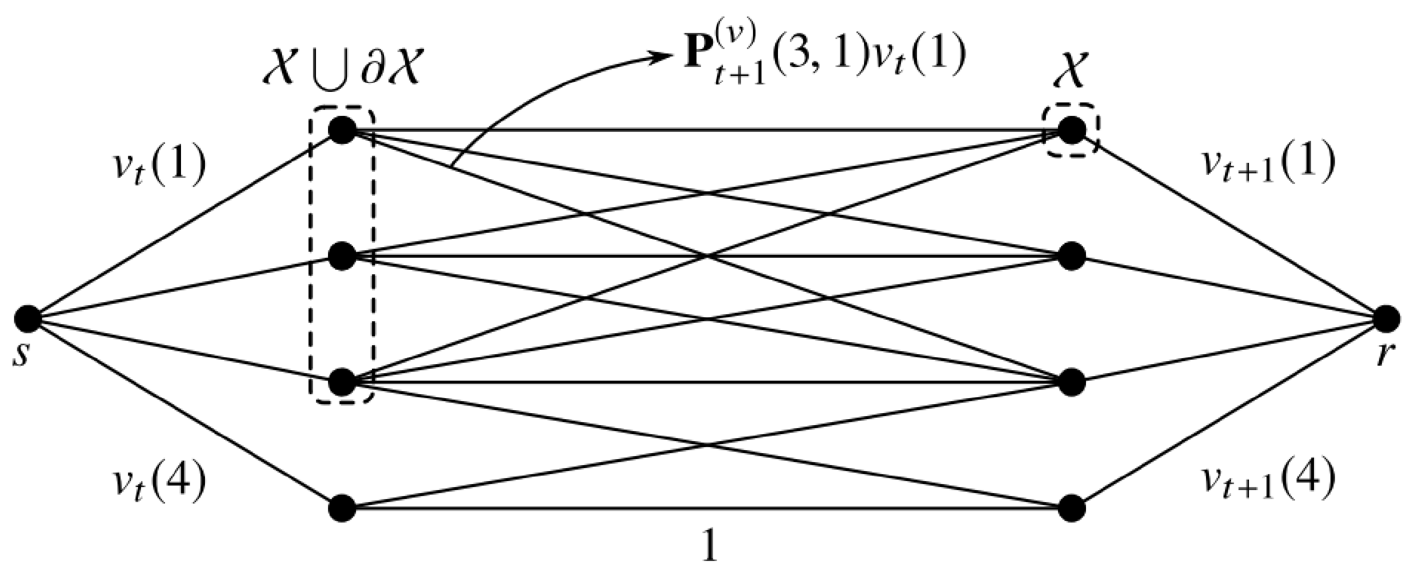
$$v_{t+1}(X) \leq v_t(X) + v_t(\partial X) \qquad \text{(locality)}$$





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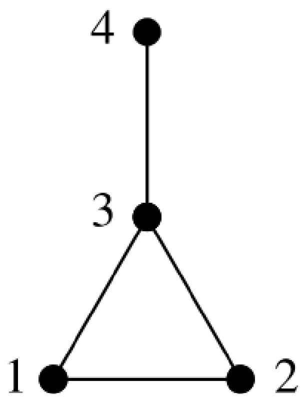
> **Stochastic bridge lemma:** Let $v_0 \xrightarrow{t} v_t$ be a local stochastic process. For all $v_0, t \geq 0$, we can construct a local stochastic transition matrix $P_{t+1}^{(v_0)} \sim G$ such that
> $$v_{t+1} = P_{t+1}^{(v_0)} v_t$$

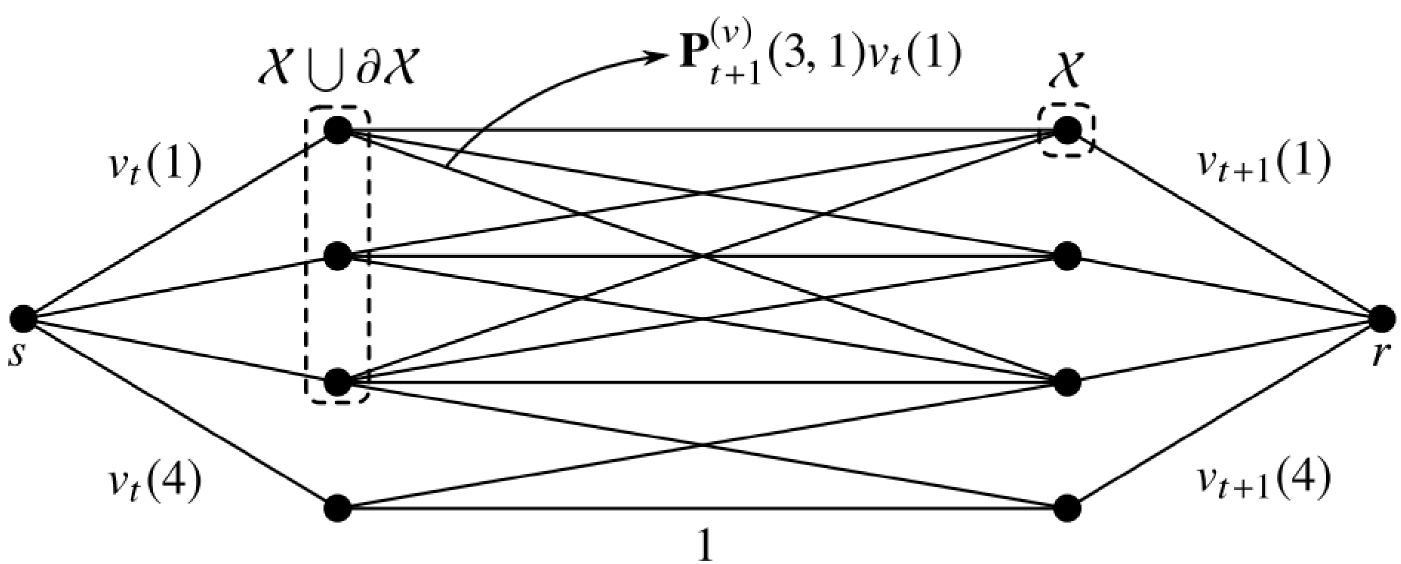
proof: max-flow min-cut argument

$$C = (1 - v_{t+1}(X)) + v_t(X) + v_t(\partial X)$$

$$v_{t+1}(X) \leq v_t(X) + v_t(\partial X) \qquad \text{(locality)}$$

$$\Rightarrow C \geq 1$$





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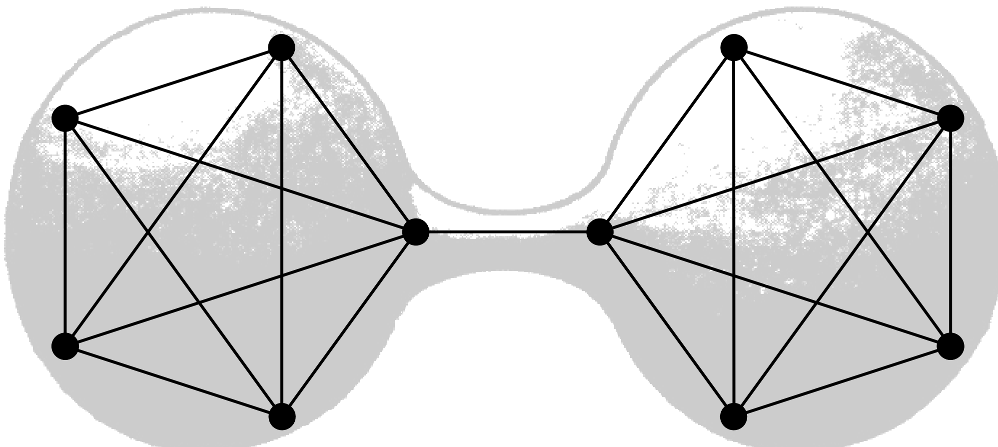

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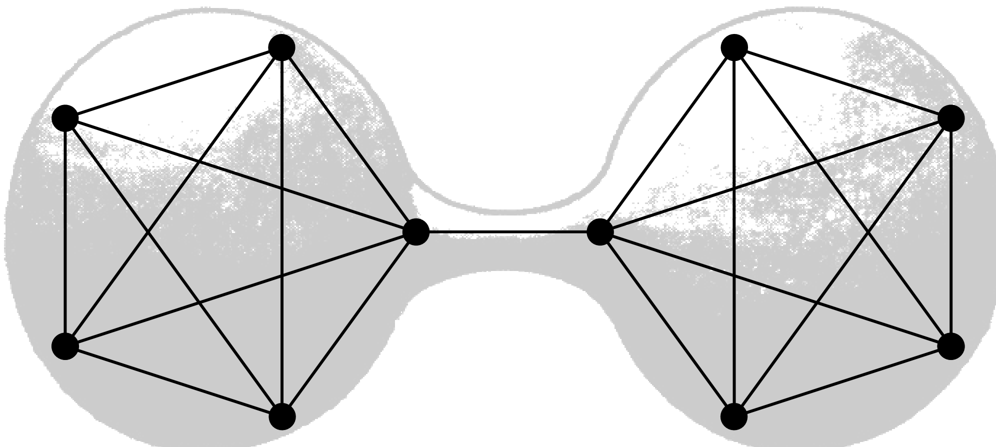

Transition rule: If $X_0 = i$ and current state $X_t = j$, go to k with probability $P_{t+1}^{(e_i)}(k, j)$.

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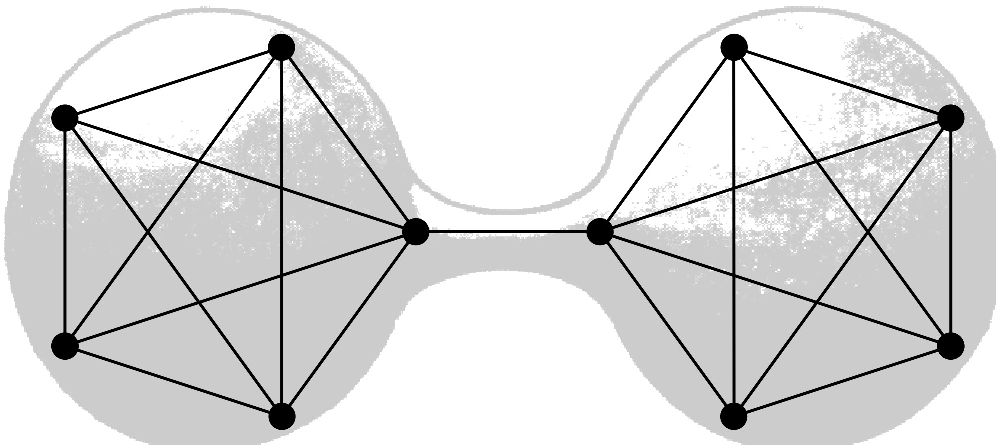

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if stochastic process is linear and local, then this transition rule simulates the process:

$$\mathbb{P}(X_t = j \mid X_0 \sim v_0) = v_t(j)$$

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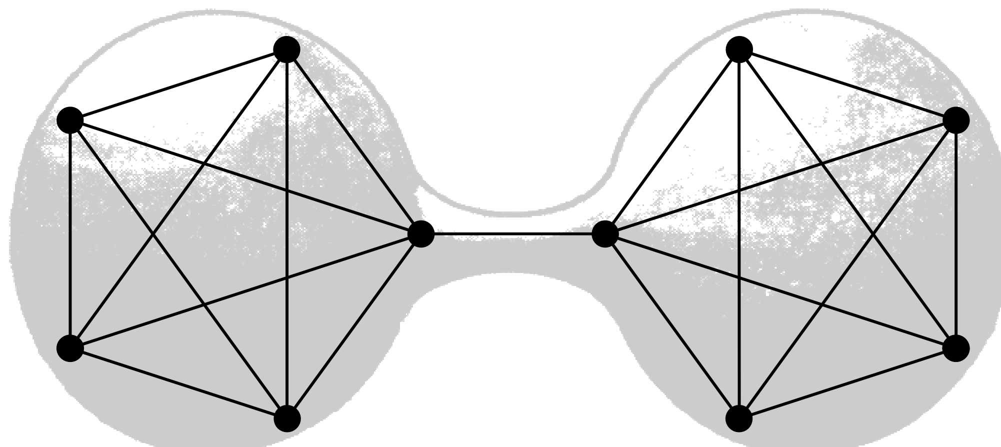

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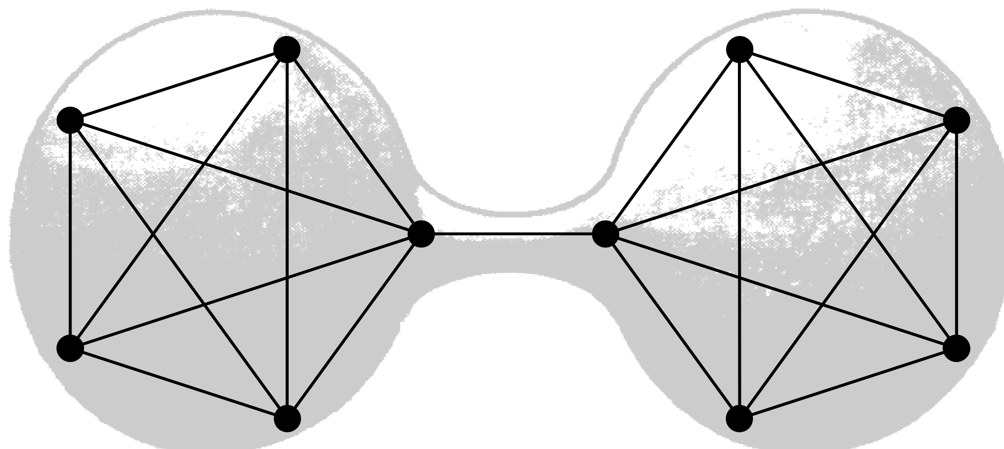

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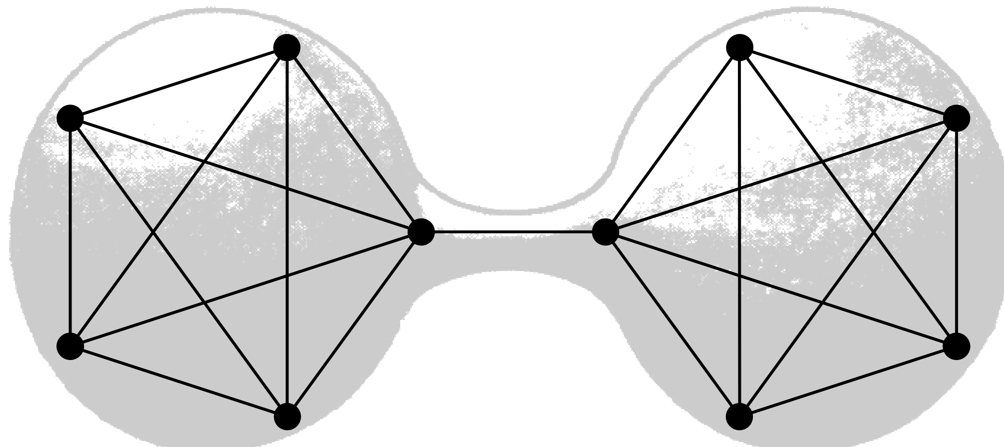

classic trick:
give walker a timer and a memory of initial state

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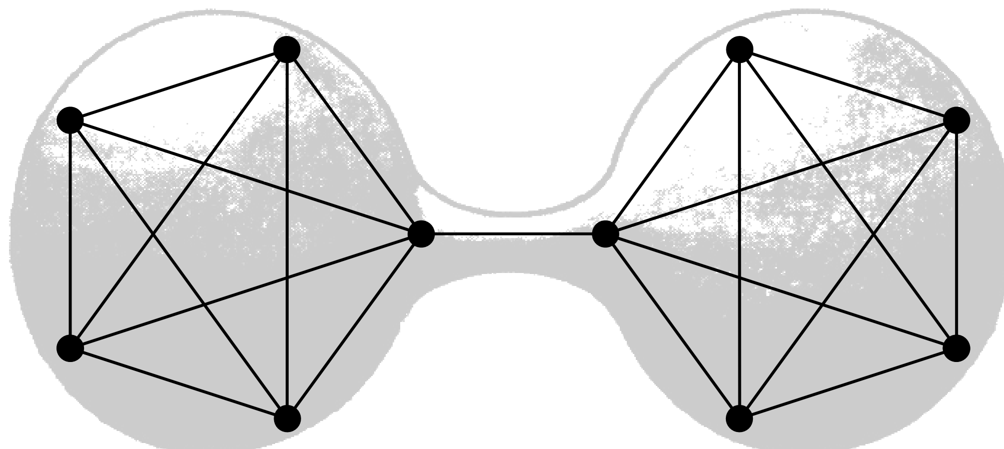

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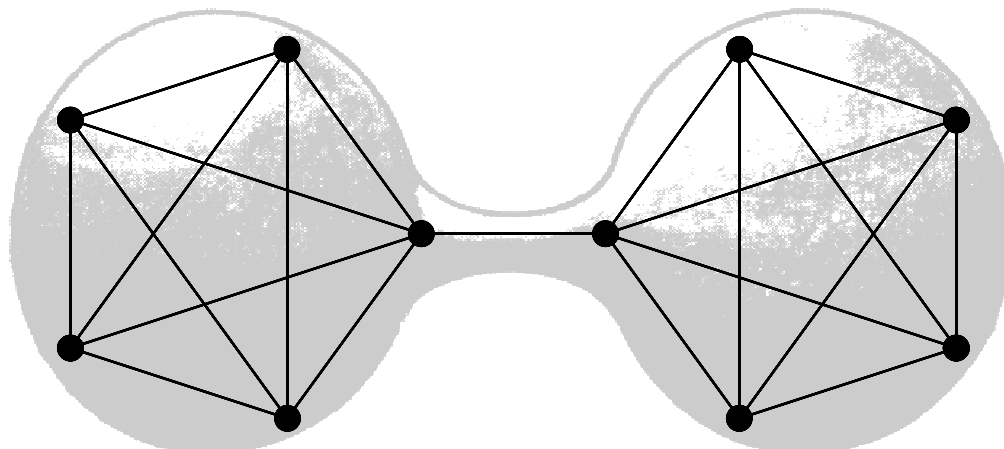

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$$e_i \otimes e_0 \otimes e_i$$
$$\downarrow$$
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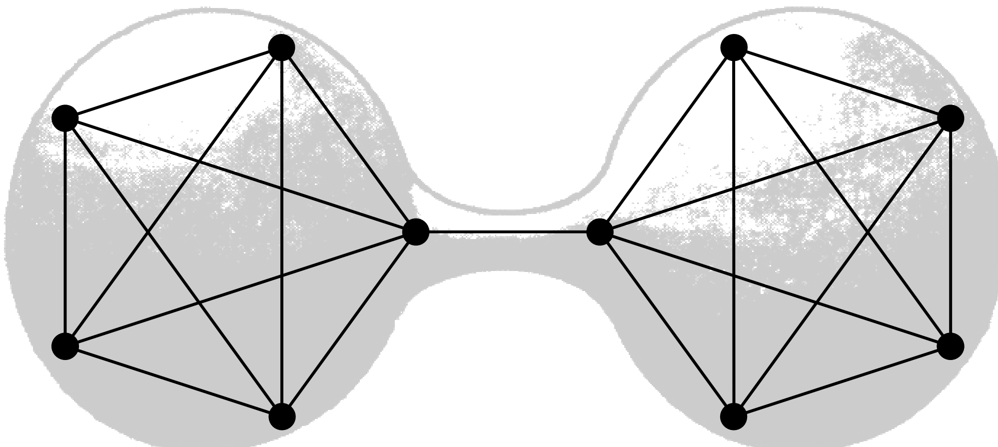
$$\downarrow$$
$$e_i \otimes e_2 \otimes P_2^{(e_i)} P_1^{(e_i)} e_i$$
$$\downarrow$$
$$\dots$$



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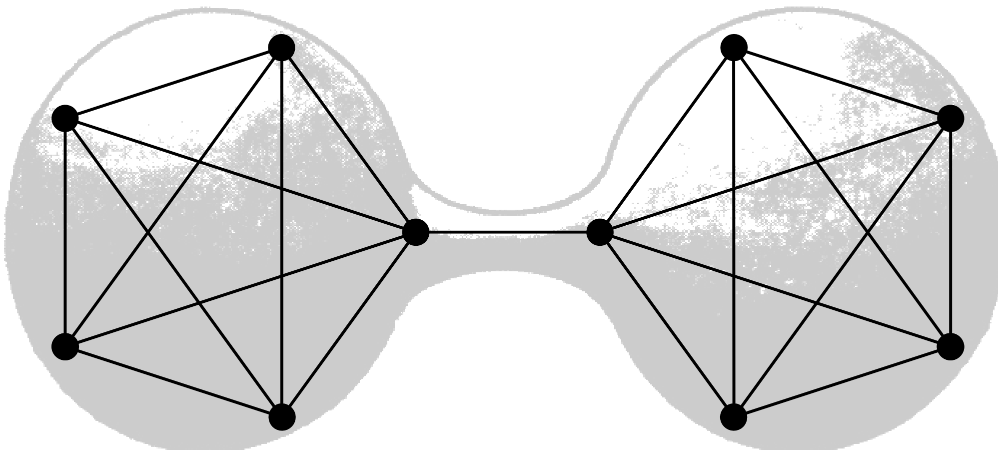

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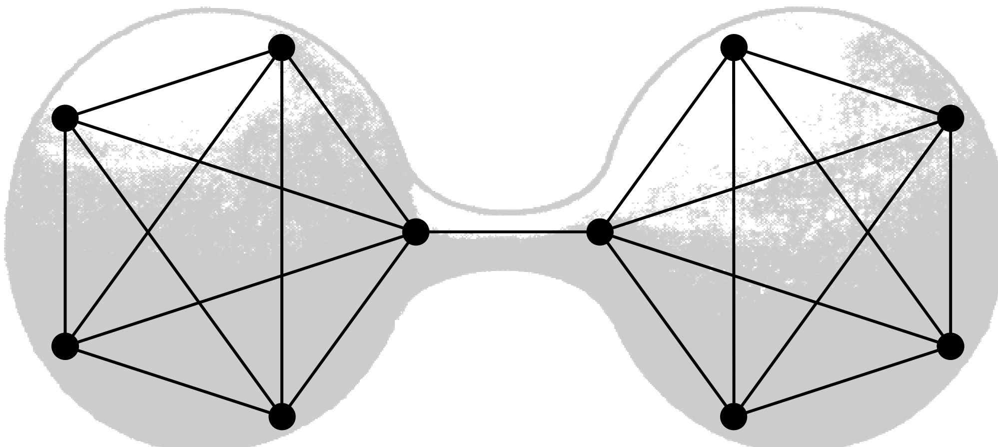

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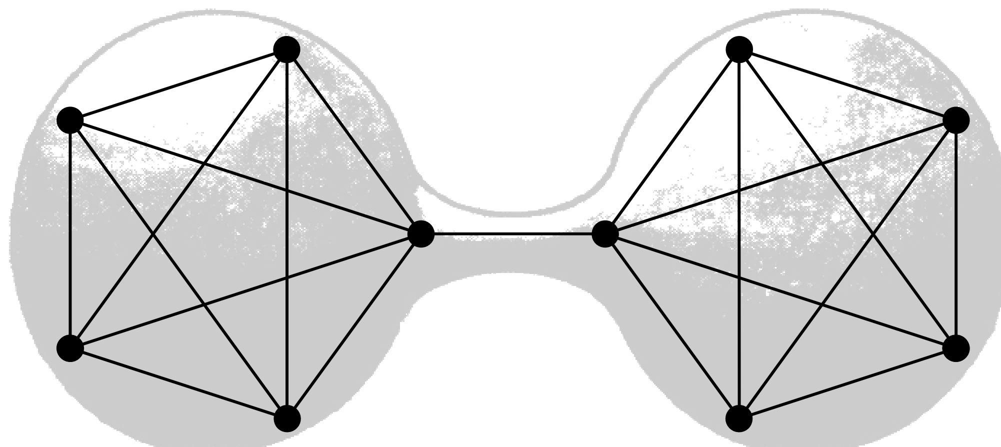

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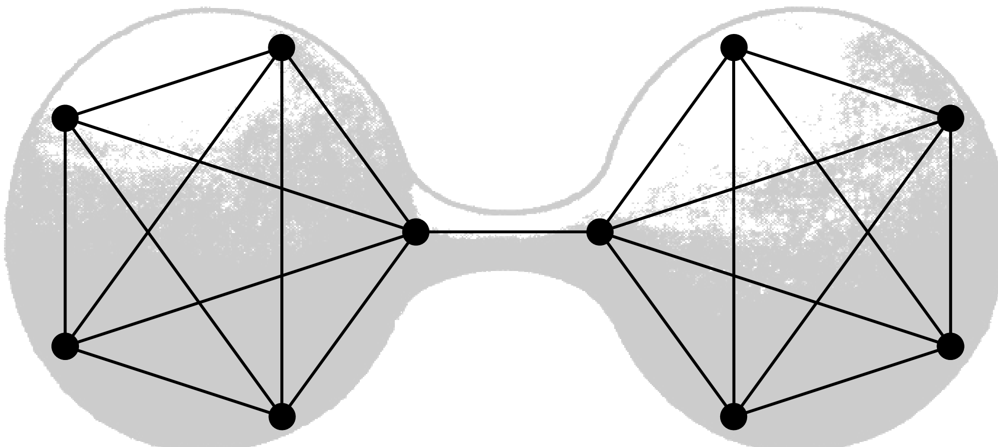

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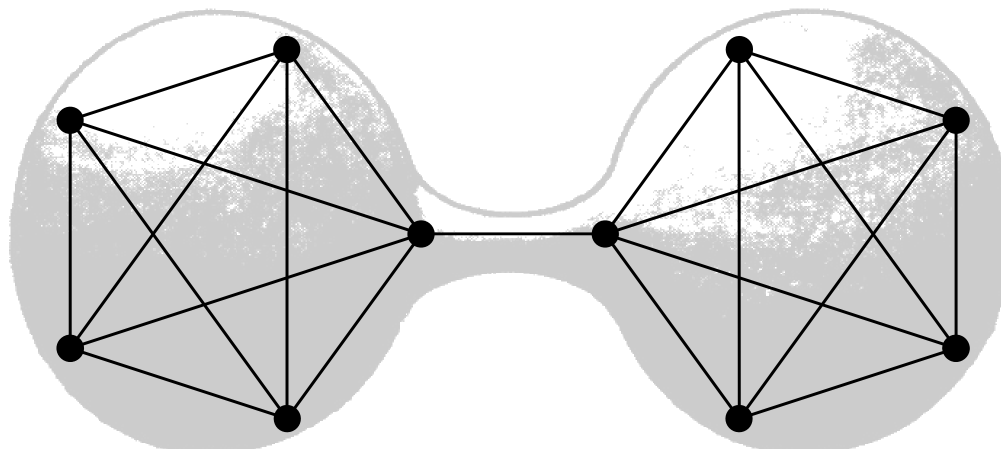
proposition:
the (asymptotic) mixing time of this amplified simulator closely relates to the (asymptotic) mixing time of the original process

2) Markov chain simulator obeys a conductance bound:



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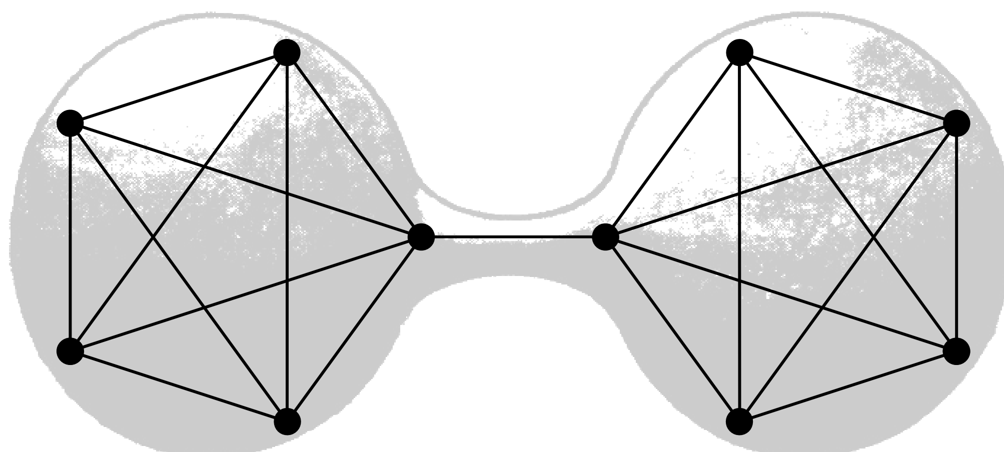
simulator is Markov chain on enlarged state space: $$\hat{\tau} \geq \frac{1}{\Phi(\hat{P})}$$



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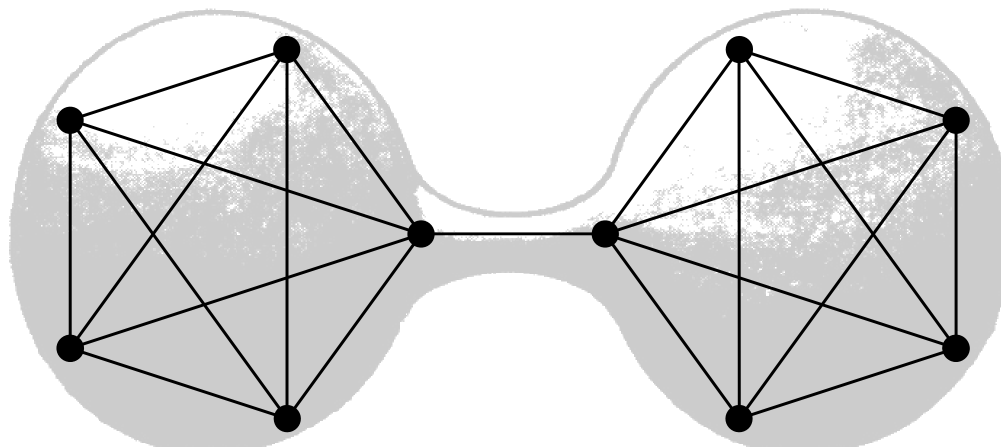

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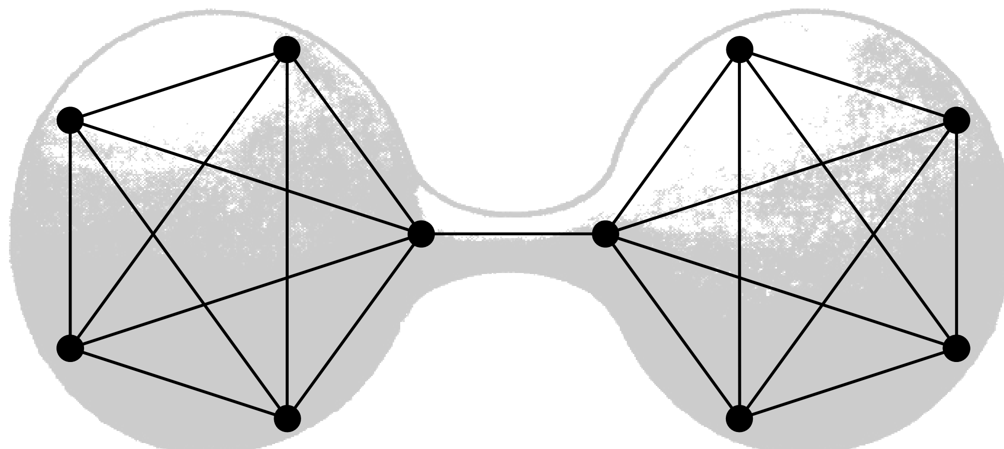

= main theorem:

any linear, local and invariant stochastic process has a mixing time

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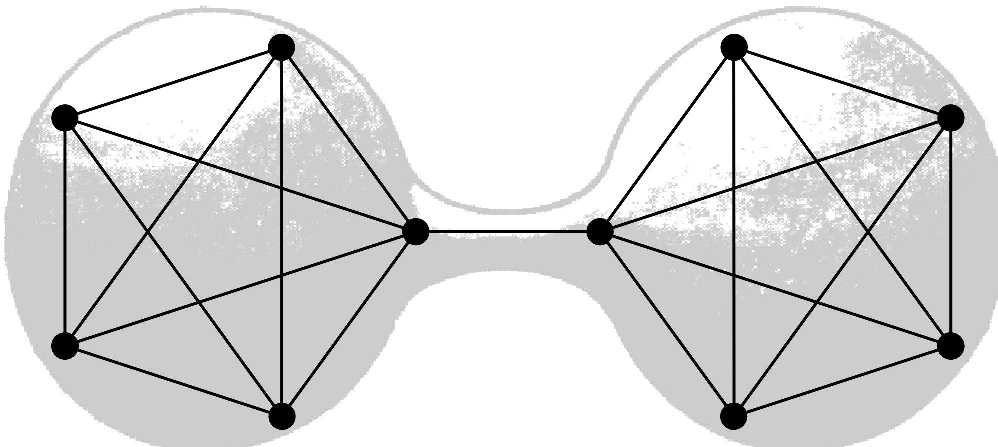

example 1: dumbbell graph

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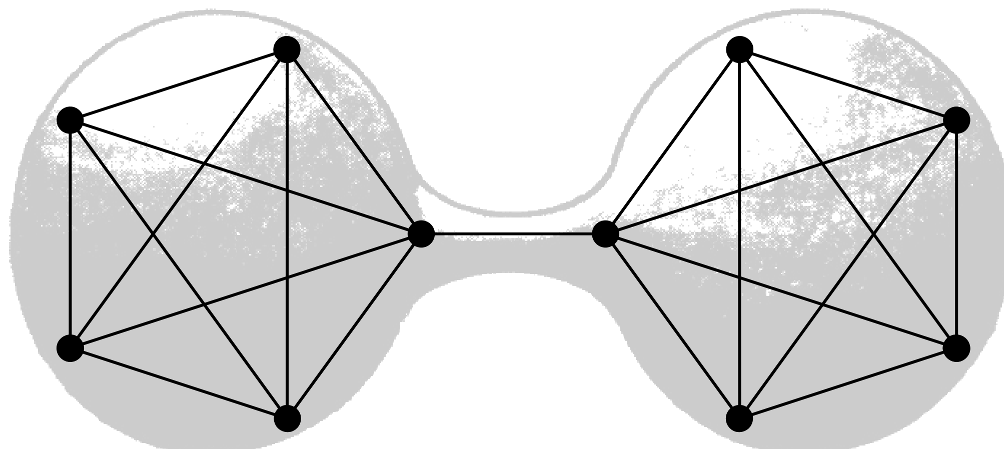

example 1: dumbbell graph

any linear, local and invariant stochastic process on the dumbbell graph has a mixing time

$$\tau \geq N$$

main theorem:

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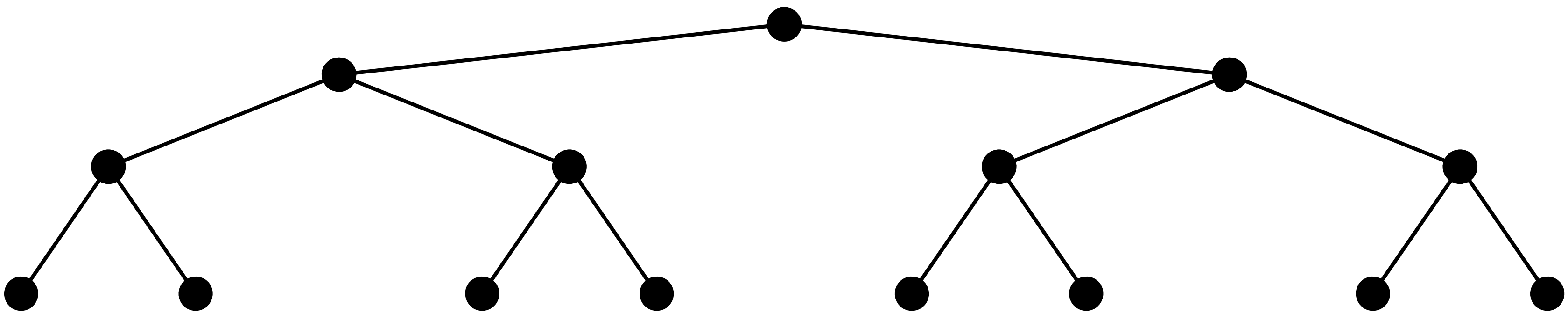
$T_{2,k}$



example 2: binary tree

main theorem:

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$$\Phi_{G,\pi} = \frac{1}{2^k}$$

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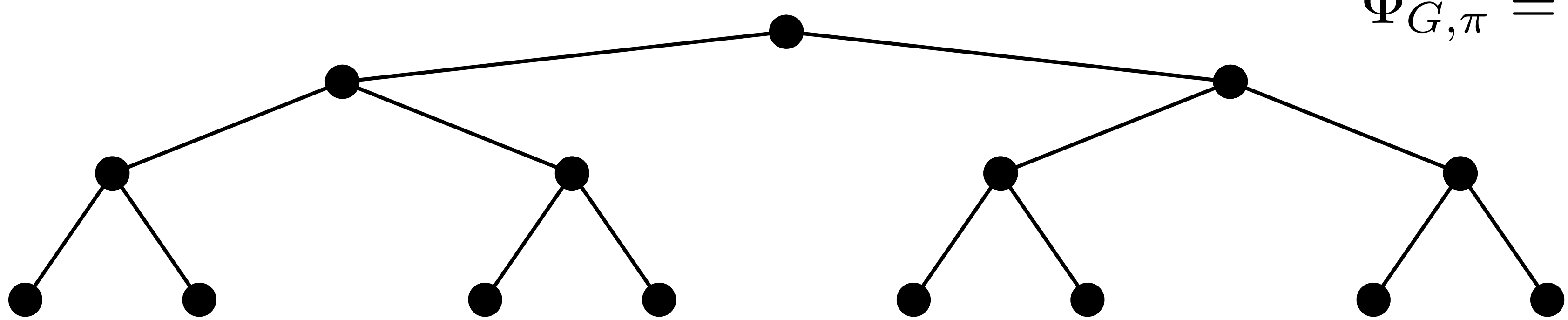

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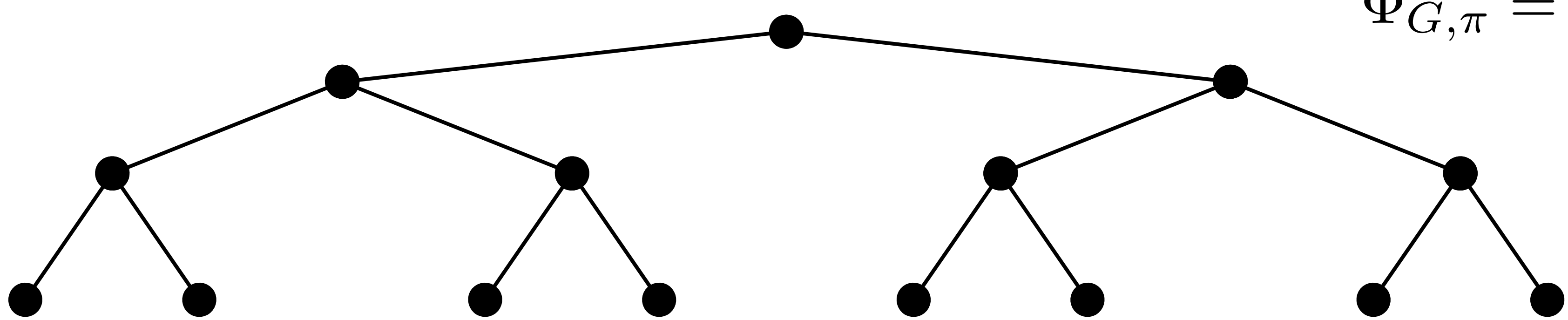

example 2: binary tree

any linear, local and invariant stochastic process on the binary tree
has the same mixing time as a random walk

$$\tau \geq 2^k = \tau_{RW}$$

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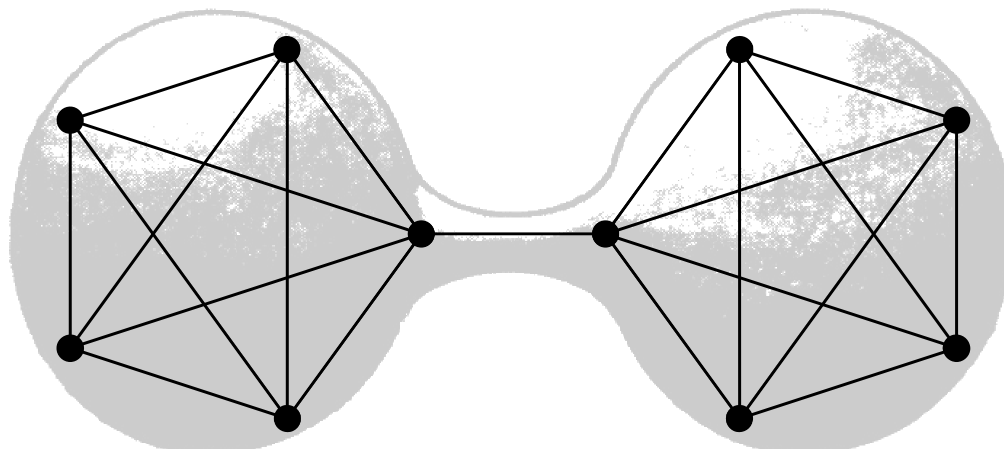

example 3: finite time convergence

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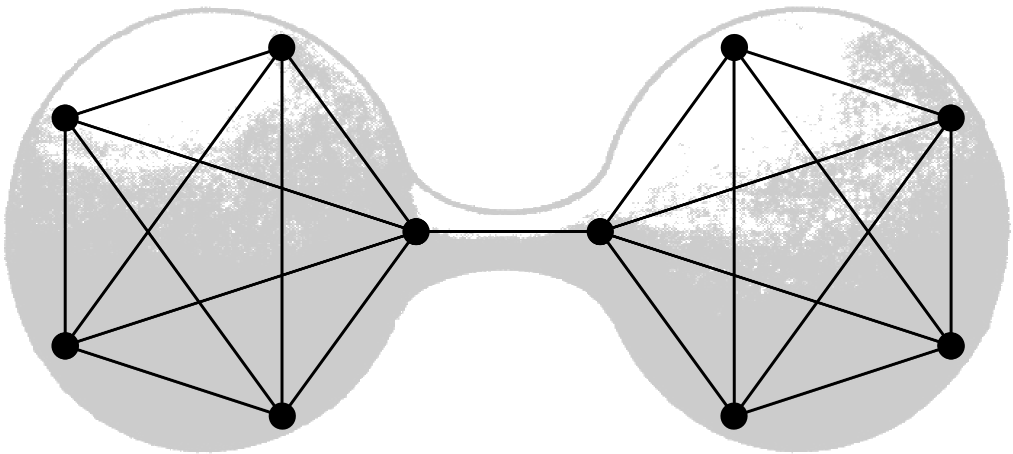

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what is the least number of local, symmetric stochastic matrices whose product has rank one?

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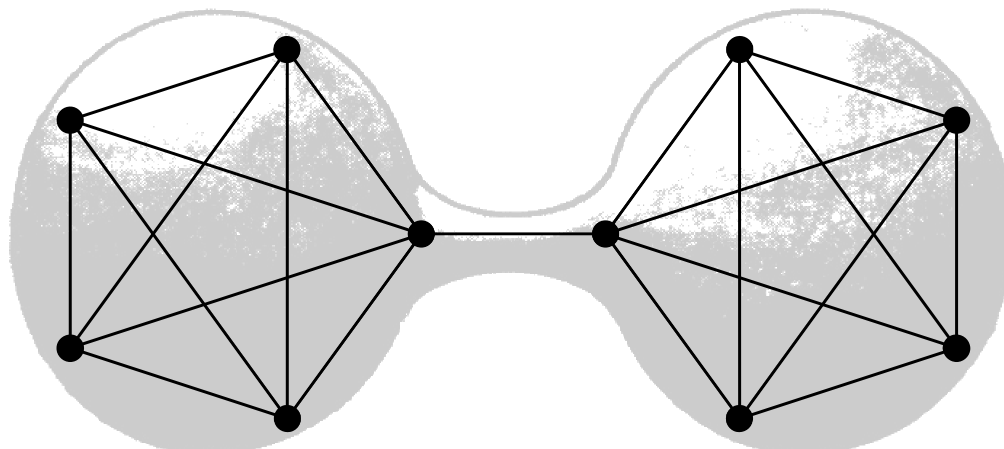

example 3: finite time convergence

what is the least number of local, symmetric stochastic matrices whose product has rank one?

= mixing time of time-inhomogeneous symmetric Markov chain

$$\geq \frac{1}{\Phi_{G,\pi}}$$

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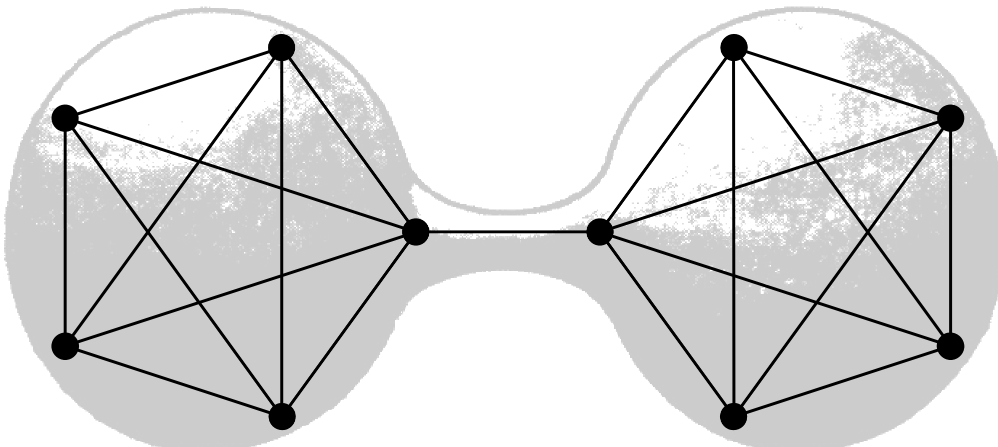

example 4: quantum walks

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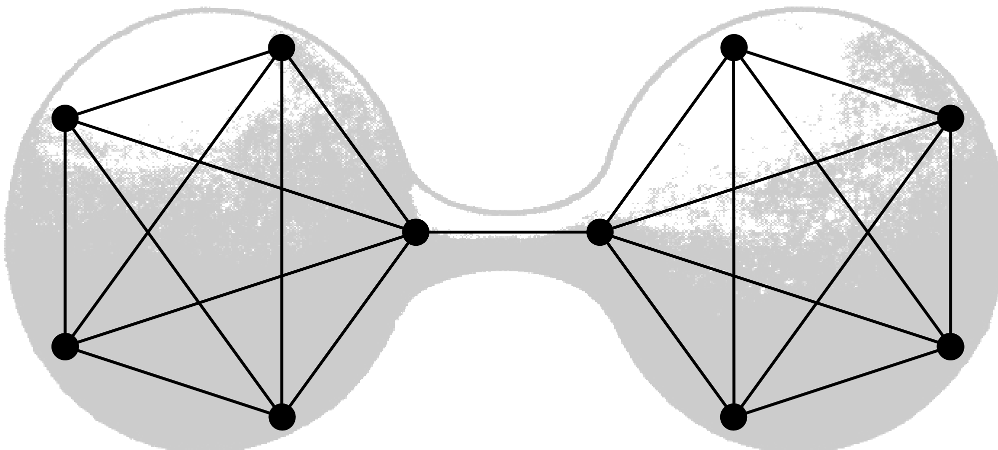

example 4: quantum walks

first bound for the mixing time of general quantum Markov chains

see details in [arXiv:1712.01609]

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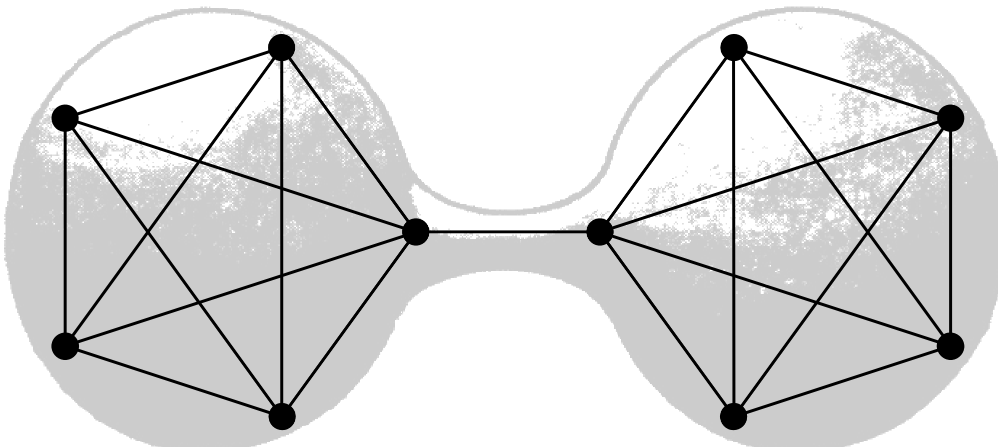

observation 1: bound is “tight”

main theorem:

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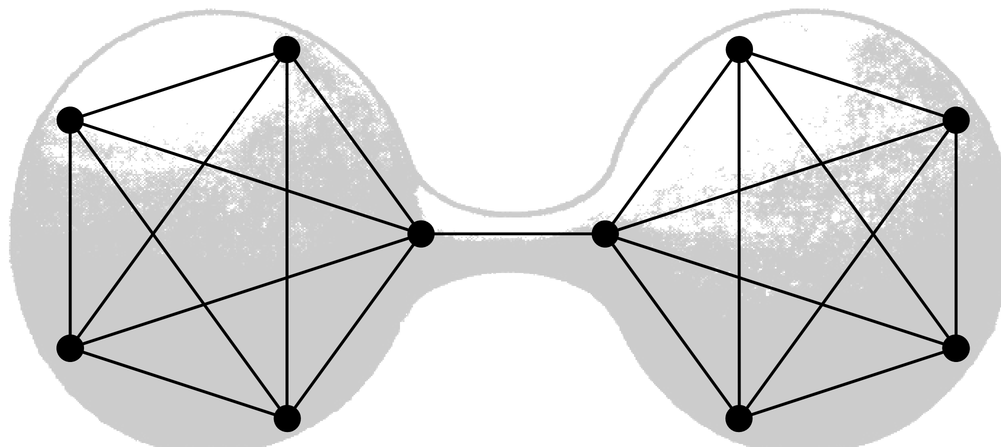

observation 1: bound is “tight”

there exists a linear, local and invariant stochastic process that has a mixing time

$$\tau \leq \frac{1}{\Phi_{G,\pi}} \log |\mathcal{V}|$$

see Chen, Lovász and Pak (STOC’99)

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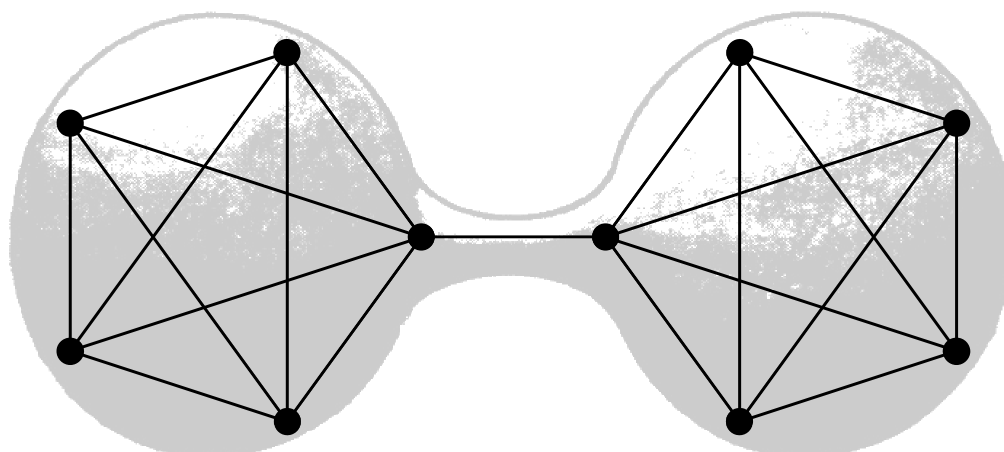

observation 2: invariance condition is necessary

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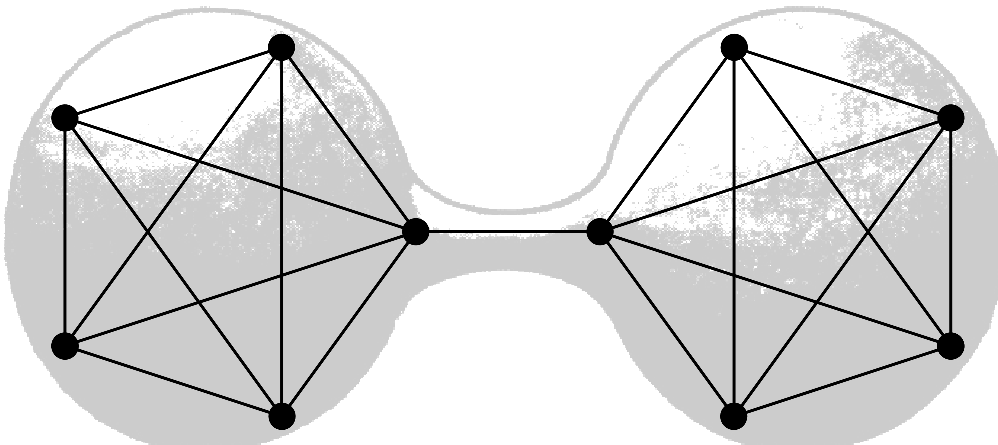

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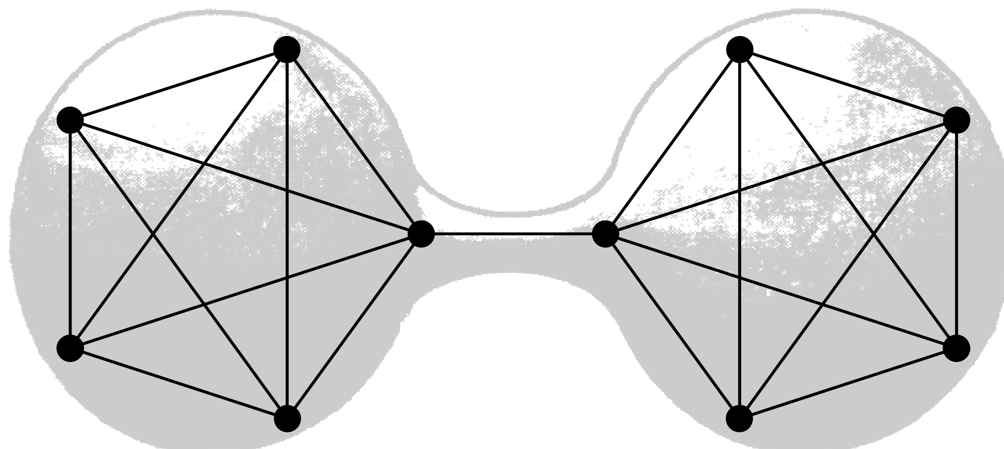

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see Pavon and Ticozzi, Journal of Math.Ph. (‘10): $\forall v, v' > 0,\ \exists \{P_t^{(v)} \sim G\}_{t=1}^{D} \text{ s.t. } P_D^{(v)} \dots P_1^{(v)} v = v'$

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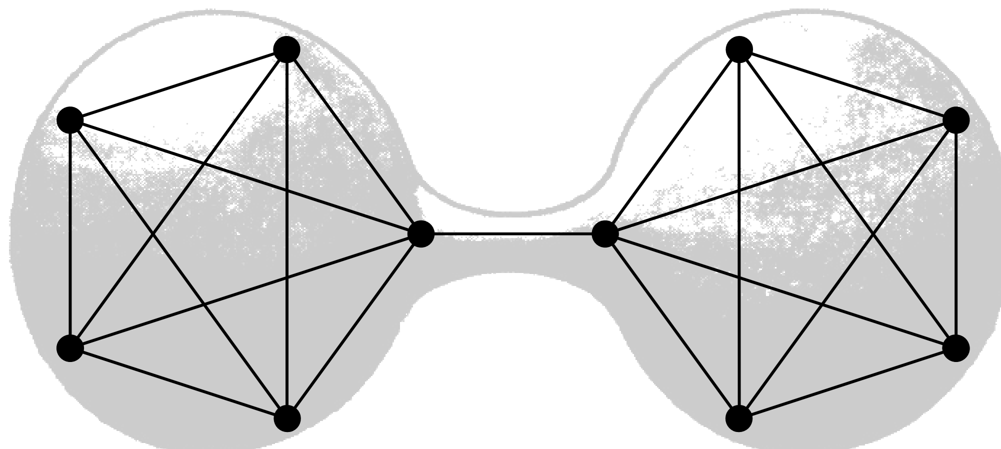

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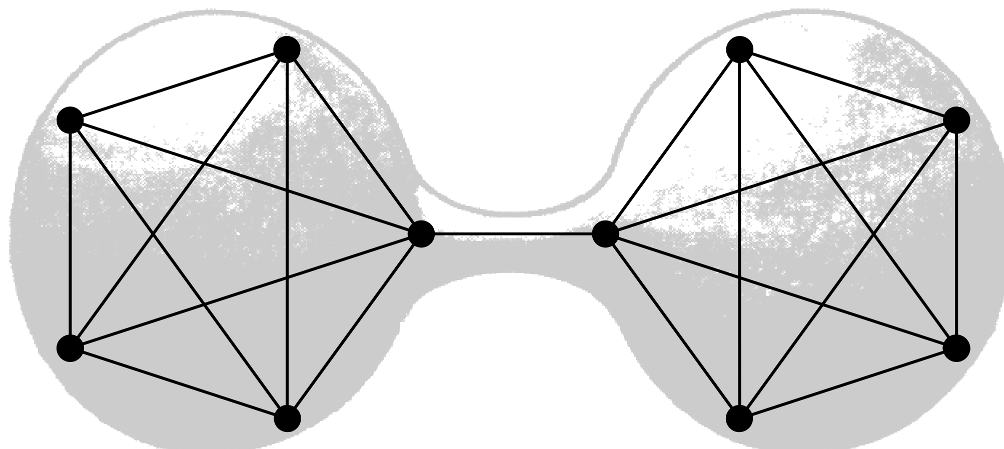

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- stronger locality form, assuming e.g. symmetry:

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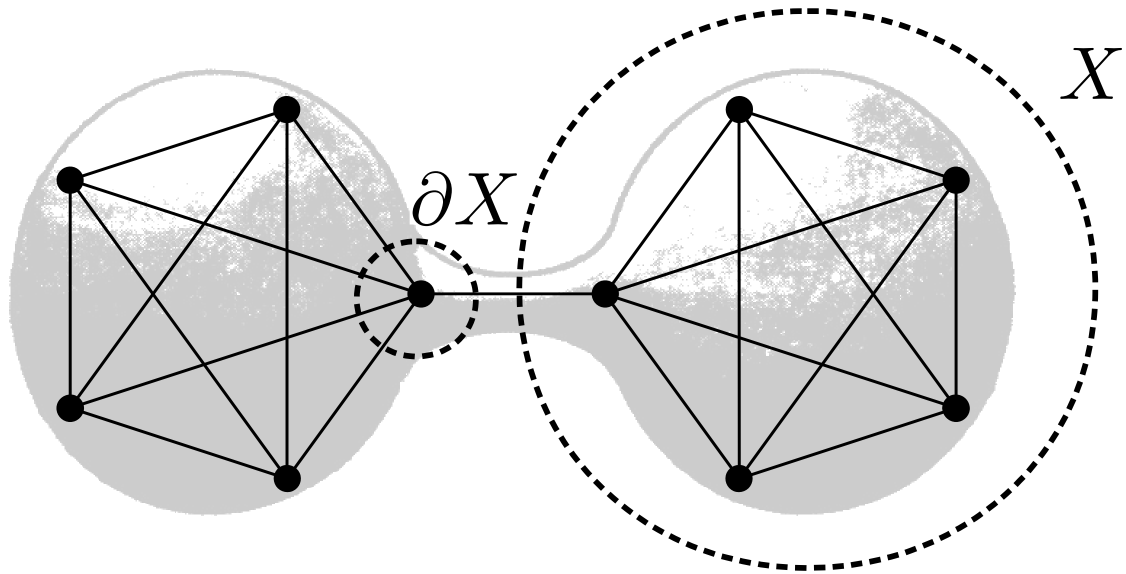

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- stronger locality form, assuming e.g. symmetry: $v_{t+1}(X) \leq v_t(X) + \frac{1}{N} v_t(\partial X)$

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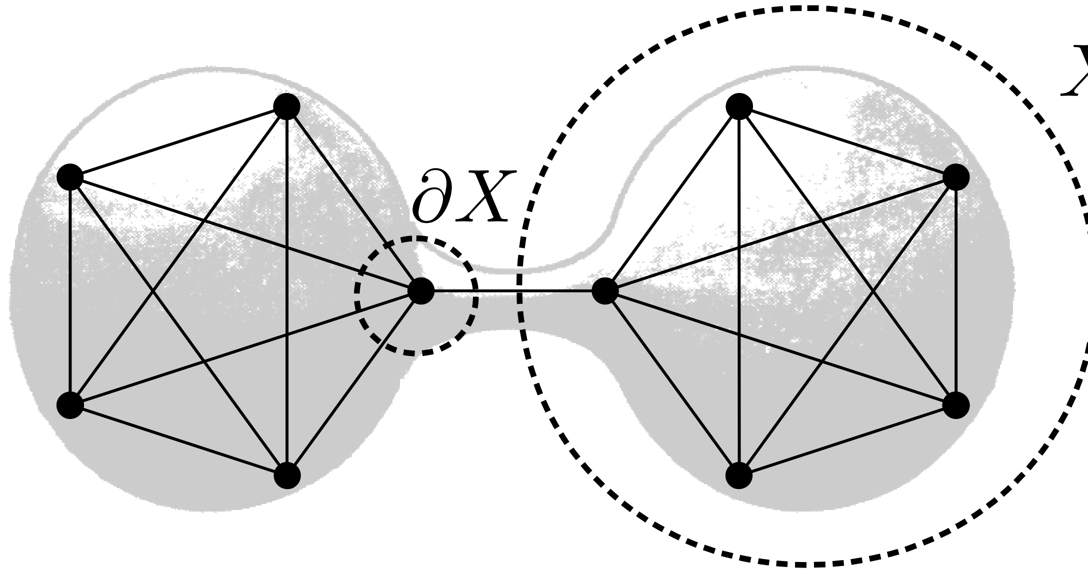

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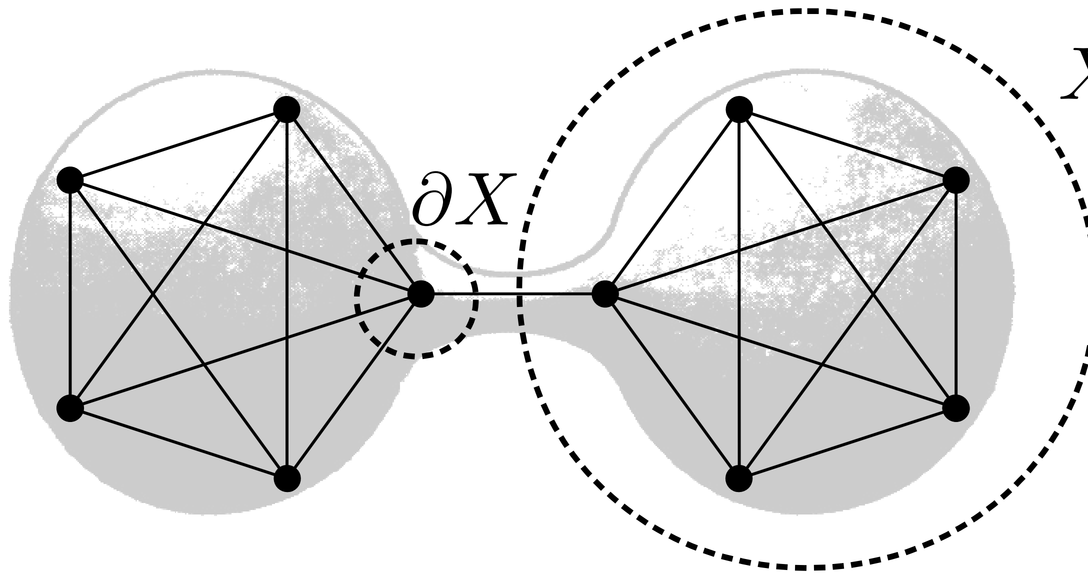

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- closed form for $\Phi_{G,\pi} = \max_{P \sim G : P\pi = \pi} \Phi(P)$