(No) Quantum space-time tradeoff for undirected STCON

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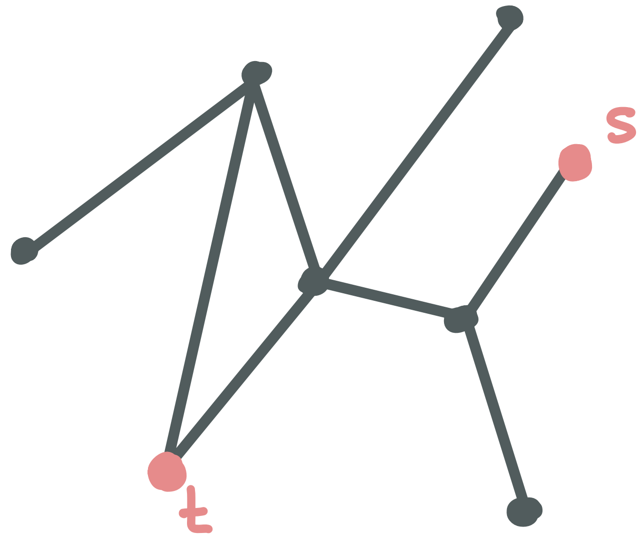
The problem

Let $G = (X, E)$ be a simple undirected graph.

$|X| = n$



Let $s, t \in X$ be two vertices

Are s and t in the same connected component of G?

Outline

- Classical approach with trade-offs
- Quantum model and context
- Quantum walk search
- Metropolis-Hastings walks

tools

- Optimal quantum algorithm for st-connectivity
- Matching lower bound
- Quantum algorithm with a trade-off

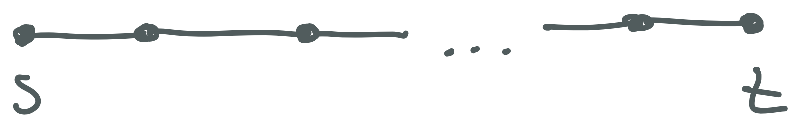
Classical approach

Goal: algorithms that give time-space trade-offs

Examples

- $\tilde{O}(n)$ space $\Rightarrow$ can remember all vertices $\Rightarrow$

  can perform BFS or DFS



or const

- $\tilde{O}(1)$ space $\Rightarrow$ can only remember one vertex $\Rightarrow$

  only see the graph locally $\Rightarrow$ can only do

  a random walk.

Assume that there is $\tilde{O}(p)$ available space

can remember p vertices

<u>Approach</u> (Broder et al. '94, Feige '95, Kosowski '12)

① Sample p pebbles, s,t are also pebbles

② Run $O(\log n)$ short Metropolis-Hastings walks from each pebble

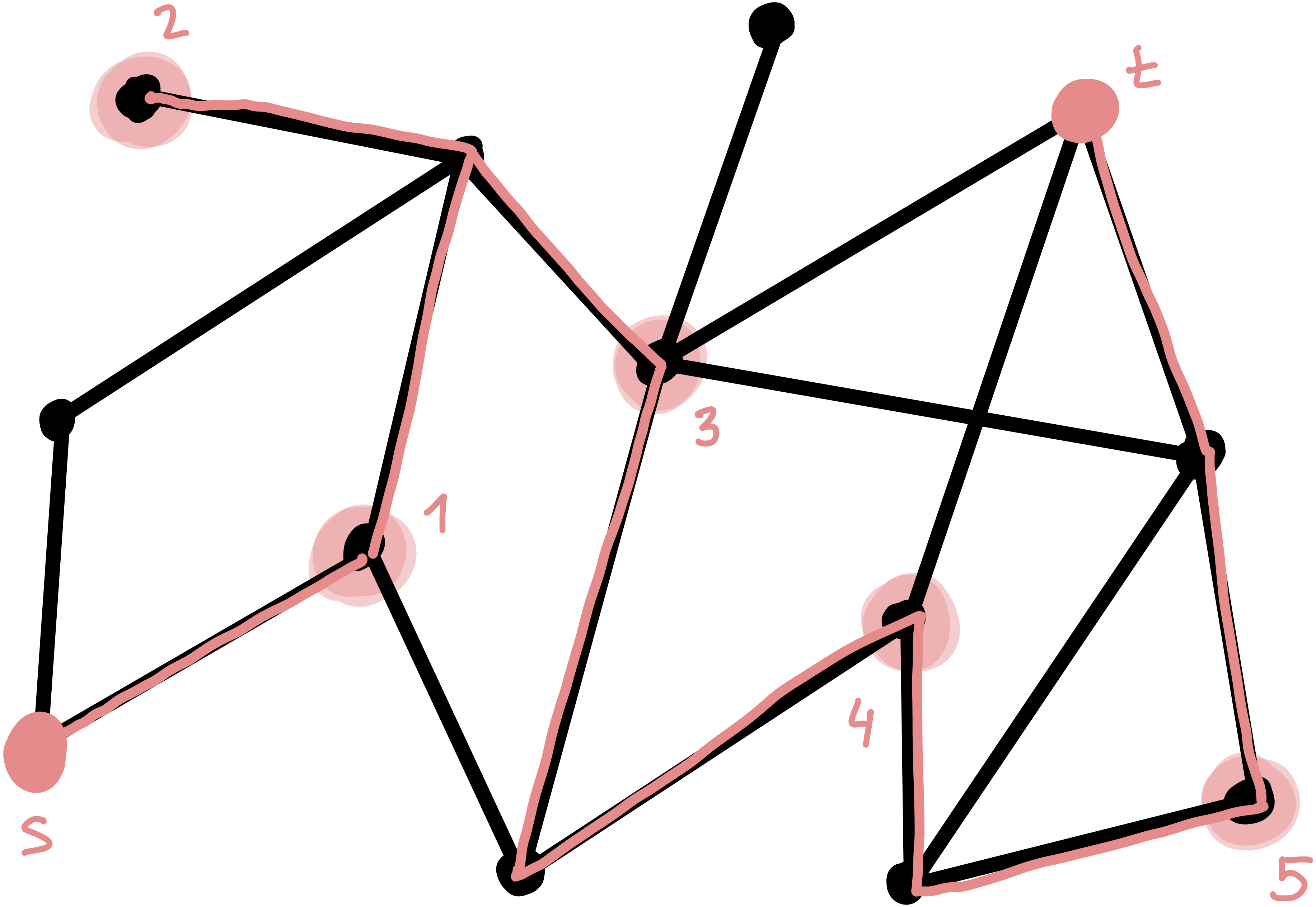
$\frac{n^2}{p^2}$

a modification of the graph

- if hit a pebble, memorize this connection (union, find)
- Check in the end if s & t are connected

Complexity: $T = \tilde{O}\left(p \cdot \frac{n^2}{p^2}\right) = \tilde{O}\left(\frac{n^2}{p}\right)$ $\quad S \leq \frac{n^2}{m}$

Trade-off: $T \cdot S = \tilde{O}(n^2)$


2
t
3
1
4
s
5

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Context

	$\text{USTCON}_{\text{mat}}$		
	Time	TS-tradeoffs	
Classical	$\widetilde{\Theta}(n^2)$	$S = O(\log(n)),\ T = \widetilde{O}(n^3/d)$	from $\text{USTCON}_{\text{arr}}$
		$S = \widetilde{O}(n),\ T = \widetilde{O}(n^2)$	BFS
Quantum	$\widetilde{\Theta}(n^{1.5})$	$S = O(\log(n)),\ T = \widetilde{O}(n^{1.5})$	[BR'12]
	$\text{USTCON}_{\text{arr}}$		
	Time	TS-tradeoffs	
Classical	$\widetilde{\Theta}(m)$	$T = \widetilde{O}(\max\{n^2/S, m\})$	Kosowski'12
Quantum	$\widetilde{\Theta}(n)$	$S = O(\log(n)),\ T = \widetilde{O}(n^{1.5})$	quantum walk
		$S = T = \widetilde{O}(n)$	[DHHM'06] adopted algorithm for CONNECTIVITY
		This work: $S = O(\log(n)),\ T = \widetilde{O}(n)$	

Model

Adjacency array:

Fixed ordering on neighbours

① Degree query $|u\rangle|0\rangle \longmapsto |u\rangle|d_u\rangle$, $u \in X$

② Neighbour query $|u\rangle|i\rangle|0\rangle \longmapsto |u\rangle|i\rangle|v_i(u)\rangle$

$$u, v_i(u) \in X,\ i \in [d_u]$$

Classically allows sampling a uniformly random neighbour

Quantum walk version:

Span(X)⊗span(X), unitary map

$$\boxed{|u\rangle|0\rangle \longmapsto \frac{1}{\sqrt{d_u}} \sum_{v \in N(u)} |u\rangle|v\rangle}$$

Quantum version of sampling a neighbour (uniformly)

can be implemented w/ quantum Adj. arr. + $\boxed{i<j \Rightarrow v_i(u) < v_j(u)}$

assumption on neighbours ordering

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Random walks: naive search algorithm

- $\omega: E \to \mathbb{R}_+$ weights on edges

$\forall u \in X$: move to $v \in X$ with probability

$$P_{uv} = \frac{\omega_{uv}}{\sum_{x \in N(u)} \omega_{ux}}$$

- $M \subseteq X$ set of marked vertices

<u>Goal</u> : find a vertex from M

1. Setup a vertex (e.g. sample from a distribution)

2. Update the vertex according to ω
3. Check if the new vertex is marked

Repeat

Complexity: $S + HT(U + C)$

$\mathbb{E}$ (# of walk steps needed to hit M)

depends on M & where we start

Quantum version

Can define quantum versions of these operations

$S: |0\rangle \longmapsto \sum_{u \in X} \sqrt{\sigma(u)}\, |u\rangle$ $\quad$ σ - distribution over X $\quad$ setup

$U: |u\rangle|0\rangle \longmapsto \sum_{v \in N(u)} \sqrt{P_{uv}}\, |u\rangle|v\rangle$ $\quad$ update

$C: |u\rangle|0\rangle \longmapsto \begin{cases} |u\rangle|0\rangle, & \text{if } u \notin M \\ |u\rangle|1\rangle, & \text{if } u \in M \end{cases}$ $\quad$ check

$\mathrm{Span}(X) \otimes \mathrm{Span}(X) \otimes \mathbb{C}^2$

Note: we need iterate this process differently in order to get speedups in the quantum case.

Quantum walk search

AGJ'19

- Let $M \subseteq X$ be a set of Marked vertices
- Let s be starting vertex, $\sigma = \delta_s$, $|\sigma\rangle = |s\rangle$

Thm

There is a quantum algorithm that finds a marked vertex with constant probability in complexity

$$\underbrace{S}_{\text{Setup}} + \sqrt{C_{s,M}}\,(\underbrace{U}_{\text{update}} + \underbrace{C}_{\text{check}})$$

up to log factors

$C_{s,M}$: $\mathbb{E}$(# steps needed to hit M starting from s and go back to u)

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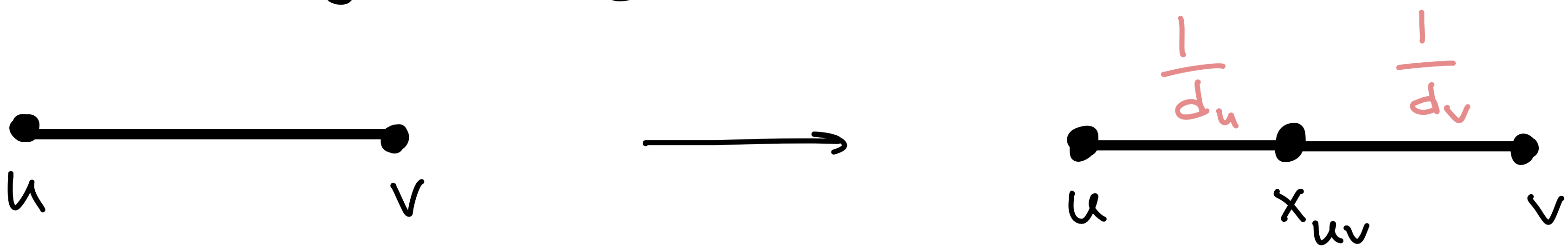
Metropolis-Hastings walks

$$P_{xy} = \frac{w_{xy}}{\sum_{z \in N(x)} w_{xz}}$$

We modify the graph G :

G'



- A walk on G' visits vertices of G on every second step $\Rightarrow$ can simulate a walk on G with a factor of 2
- Important property: hitting time is $O(n^2)$ Kosowski '12

$$\max_{u,v \in G} \mathbb{E}(\#\text{ steps to hit } v \text{ starting from } u)$$

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Single walk quantum algorithm

- Let $\sigma = \delta_s$, $|\sigma\rangle = |s\rangle$, $M = \{t\}$

- Run the quantum walk search algorithm with the Metropolis-Hastings modification

Input model:

$$|0\rangle|u\rangle \mapsto \frac{1}{\sqrt{d_u}} \sum_{v \in N(u)} |v\rangle|u\rangle$$

Metropolis-Hastings

$$C_{s,t} = H_{st} + H_{ts} \in O(n^2)$$

Markov property

small small

Complexity: $T = \tilde{O}(S + \sqrt{n^2}(U + C)) = \tilde{O}(S + n(U + C))$

need to argue

$S = O(\log n)$ optimal

Metropolis-Hastings walk Quantum implementation

Usual quantum walk assumption

$$W : |u\rangle|0\rangle \longmapsto \frac{1}{\sqrt{d_u}} \sum_{v \in N(u)} |u\rangle|v\rangle \qquad \forall u \in X$$

+

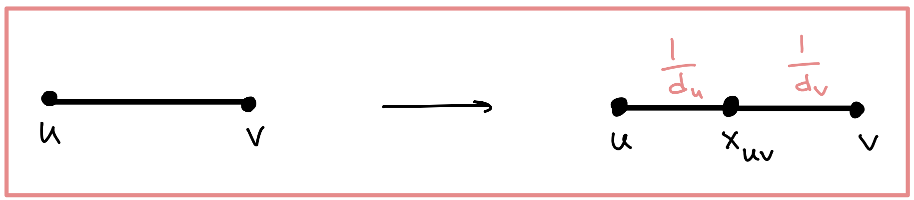
degree queries : $O_D : |0\rangle|u\rangle \longmapsto |d_u\rangle|u\rangle$

Need to implement: (Szegedy quantum walk operator)

$$U : |x\rangle|0\rangle \longmapsto \sum_{y \in N(x)} \sqrt{P_{xy}}\, |x\rangle|y\rangle \qquad \forall x \in X'$$

$$P_{xy} = \frac{w_{xy}}{\sum_{z \in N(x)} w_{xz}}$$

- For vertices of G nothing changes
- For x_{uv} rotate an auxiliary qubit controlled on degrees



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Lower bound: Parity → st-connectivity reduction

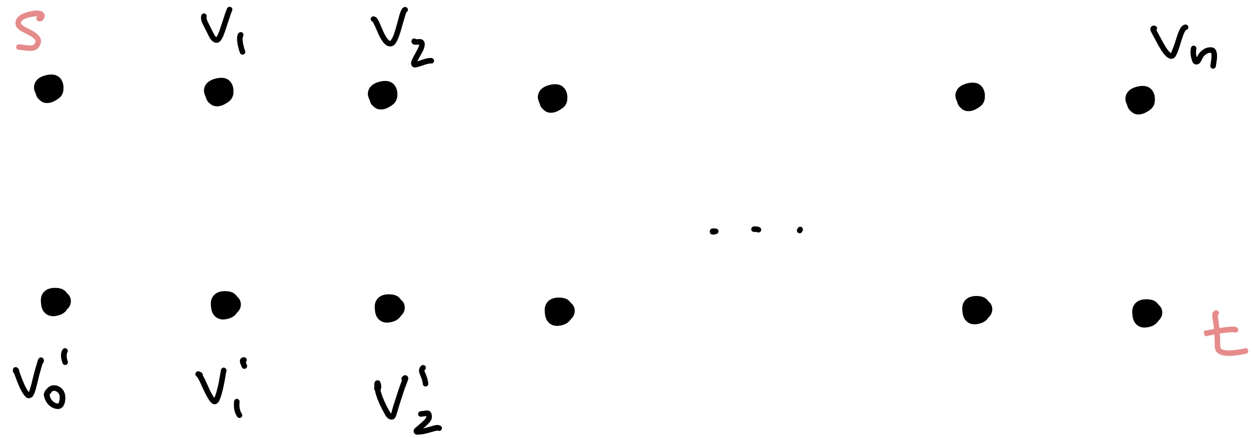
Parity problem: for $x \in \{0,1\}^n$ output $\bigoplus_{i=0}^{n-1} x_i$

BBC+ '01, FGGS '98

Lemma Quantum query complexity of parity is $\Omega(n)$.

Given a parity input $x \in \{0,1\}^n$,
we want to build an st-connectivity input.

Vertices



Edges

$$s \text{ and } t \text{ are connected} \iff \bigoplus_{i=0}^{n-1} x_i = 1$$

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About quantum trade-offs

- We saw that we don't need additional space for the general case
- Idea: look at a special case and try to find a trade-off
- Assumption: spectral gap $\geq \delta$

$\Rightarrow$ mixing time $\in \tilde{O}\left(\frac{1}{\delta}\right)$

time needed to sample from stationary distribution

Quantum algorithm with a trade-off for the case of bounded spectral gap

Algorithm

$S = \tilde{O}(p)$

(1) Run p classical random walks of length $\frac{1}{\delta}$ from each s & t

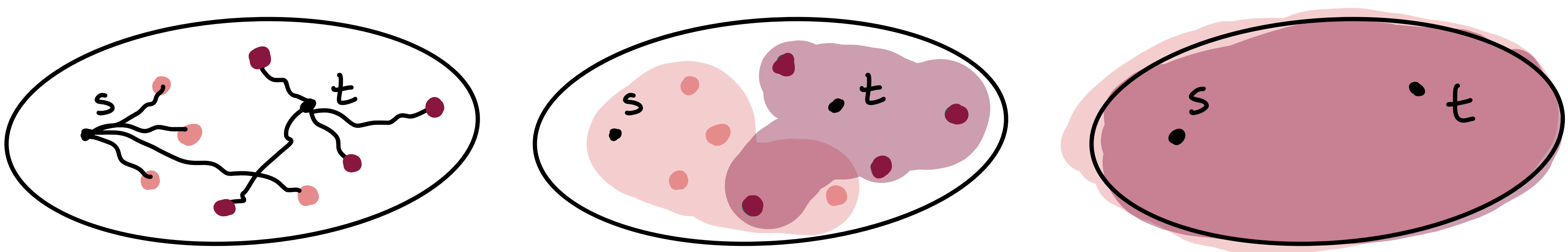
↳ sets M & L known to be connected to s & t

endpoints of walks

(2) Prepare superpositions over the connected components of s & t (inverse quantum walk search, Apers'19)

(3) SWAP test

Ignoring constants and logs



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Ignoring constants and logs

Time : $\frac{S}{\delta} + \sqrt{\frac{n}{\delta S}}$

classical walks

$\frac{1}{\sqrt{\pi(M)\delta}}$, complexity of step 2

(i.e. $\pi(M), \pi(L) \sim \frac{S}{n}$)

Decreases for $S \in \Omega(\log n)$ until $S \in O((n\delta)^{1/3})$

Thank you for your attention!