QUADRATIC SPEEDUP FOR SPATIAL SEARCH WITH CTQW

Simon Apers

(CNRS, Paris)

with

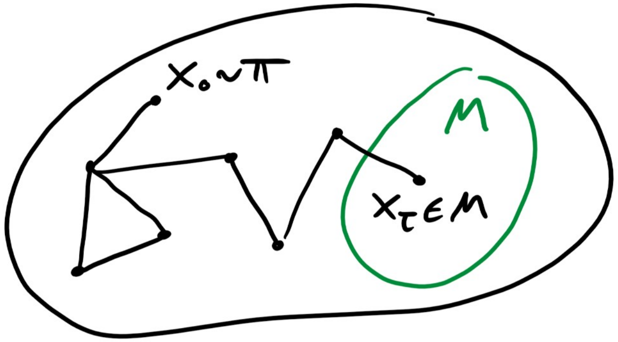
S. Chakraborty,

L. Novo & J. Roland

in

TQC '22.

* <u>BACKGROUND:</u>



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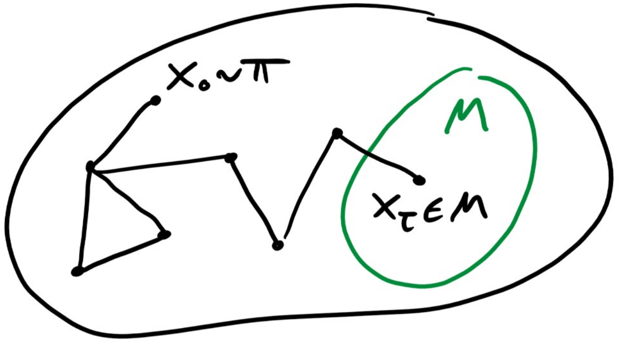

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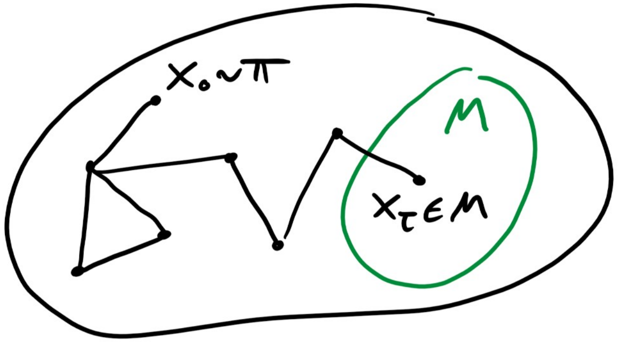

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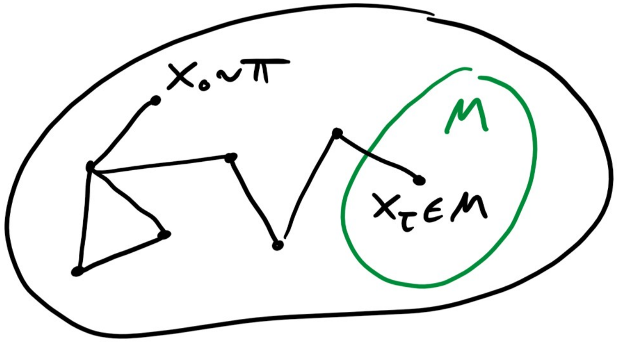

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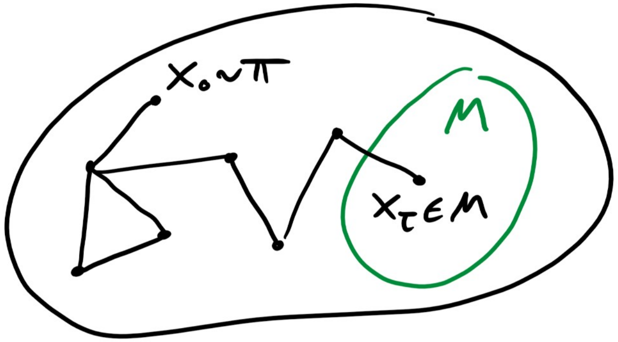

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"HITTING TIME":

- marked elements $M \subseteq V$
- $\tau(M) = \min\{t \geq 0 \mid X_t \in M, X_0 \sim \pi\}$
- $\mathbb{E}[\tau(M)] = HT(M)$

MAIN RESULT:

Starting from

can find marked element in M by running

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e.g., Childs-Goldstone '04

"simple" QW Hamiltonian

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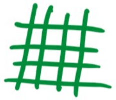

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quadratic speedup for lattices

"simple" QW Hamiltonian

but not for general graphs!

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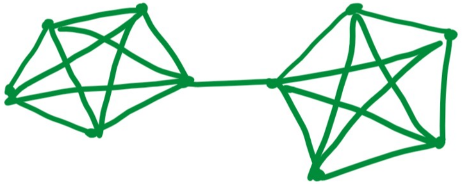

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*HURDLE 1: <u>variety of CTQWs</u>

alternative: Somma-Ortiz '10

"coined" QW Hamiltonian H_P s.t. $H_P^2 (|\psi\rangle \otimes |0\rangle)$

$= \left((\mathbb{1} - P^2) |\psi\rangle \right) \otimes |0\rangle$

à la Szegedy

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[AGJK'21] builds on discrete "quantum fast-forwarding" [Apers-Sarlette '19]

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main technique = "QUANTUM GAUSSIAN FILTER"

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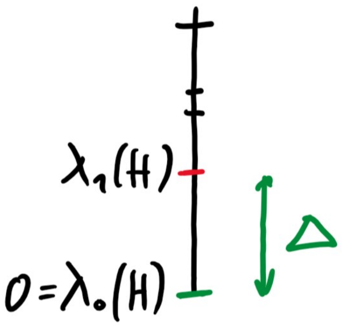

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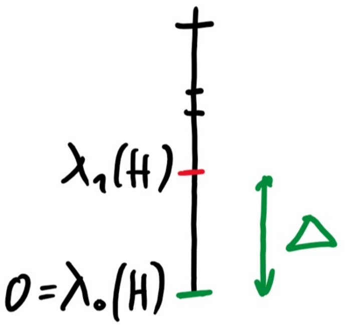

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$$e^{-itH} \rightsquigarrow e^{-tH} = \sum e^{-t\lambda_j(H)} |v_j\rangle\langle v_j|$$

$$\approx |v_0\rangle\langle v_0| \quad \text{for } t \gg \frac{1}{\Delta}.$$

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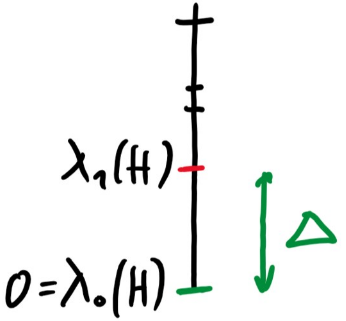

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For any $t \geq 0$, can implement

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difference w.r.t. this work:
we use filter to simulate <u>dynamics</u>

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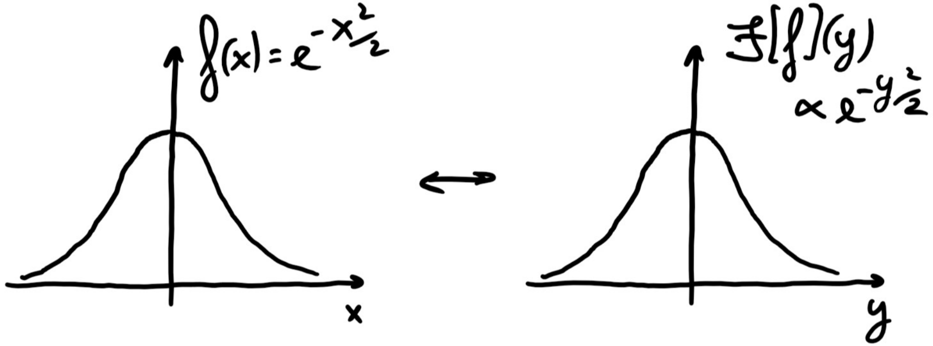

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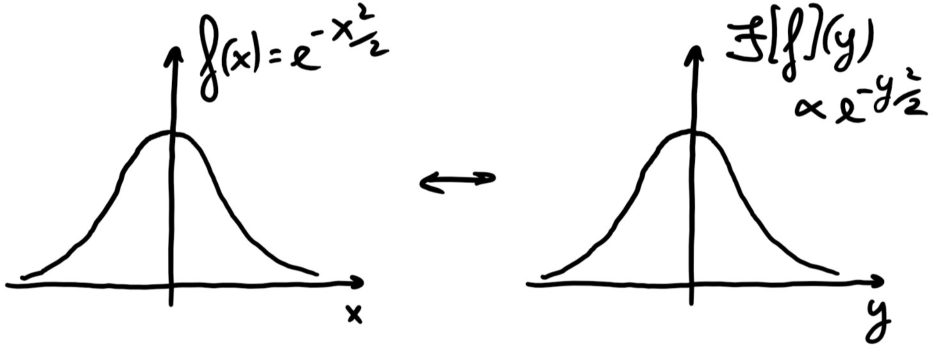

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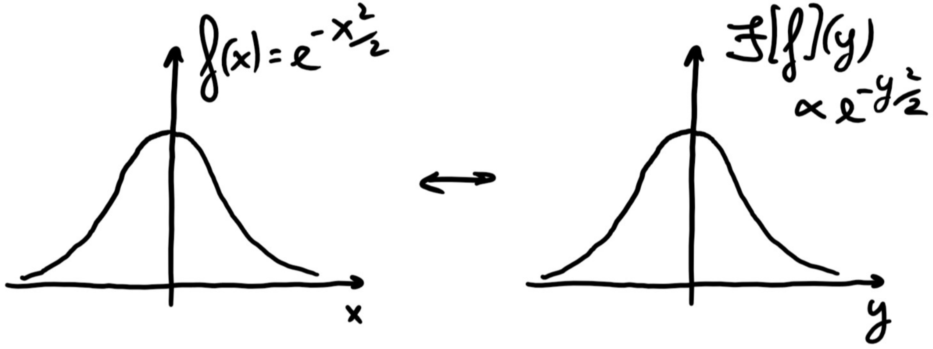

i.e., $e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int dy \, e^{-y^2/2} e^{-ixy} \quad \xrightarrow{x = \sqrt{2t} H} \quad e^{-tH^2} = \frac{1}{\sqrt{2\pi}} \int dy \, e^{-y^2/2} e^{-iy\sqrt{2t} H}$

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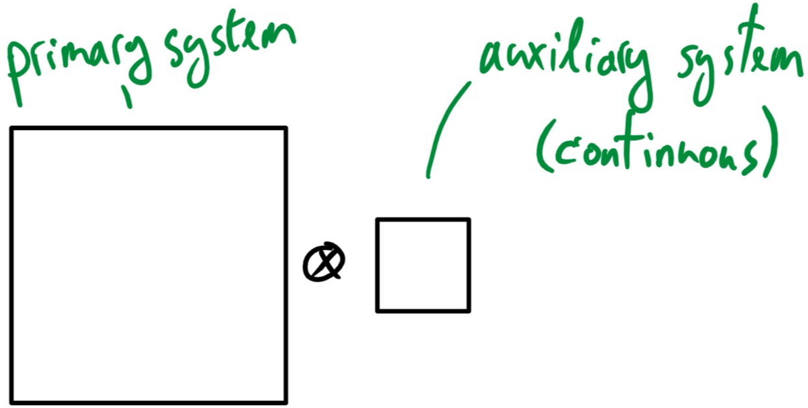

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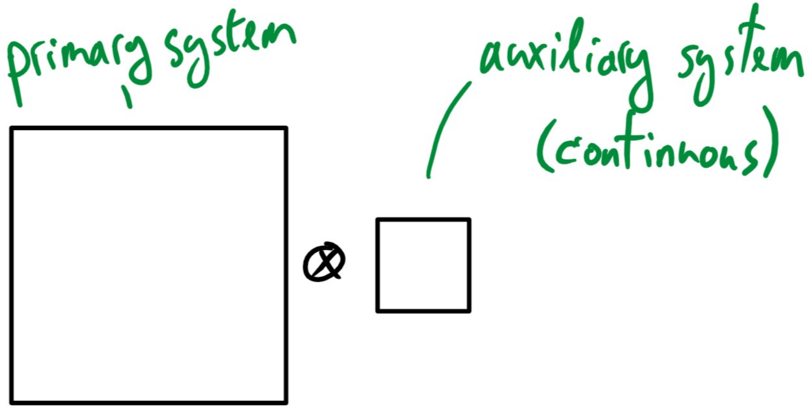

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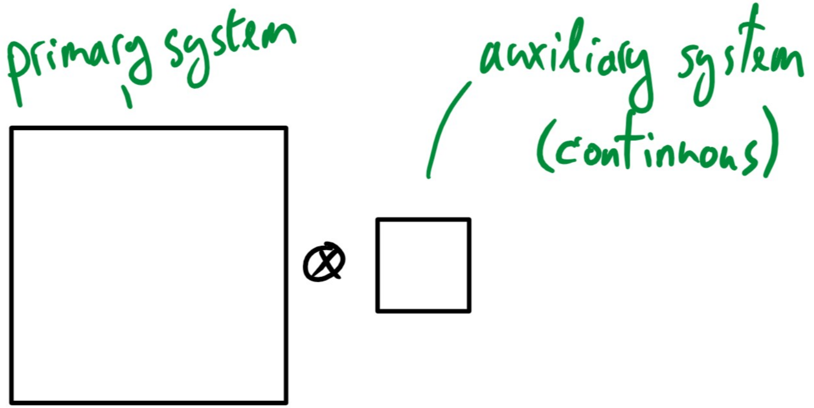

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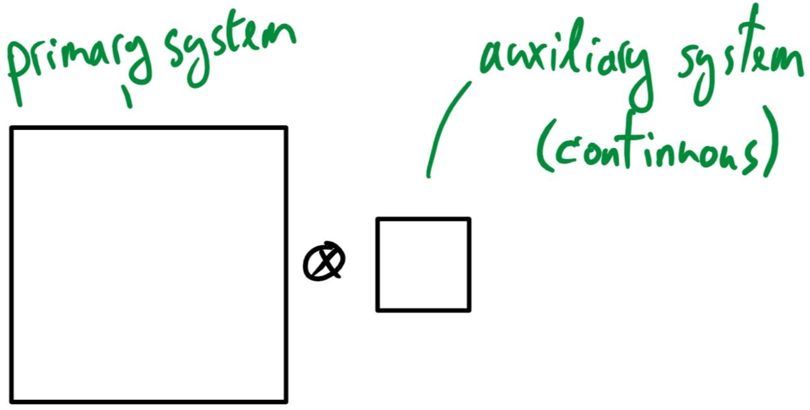

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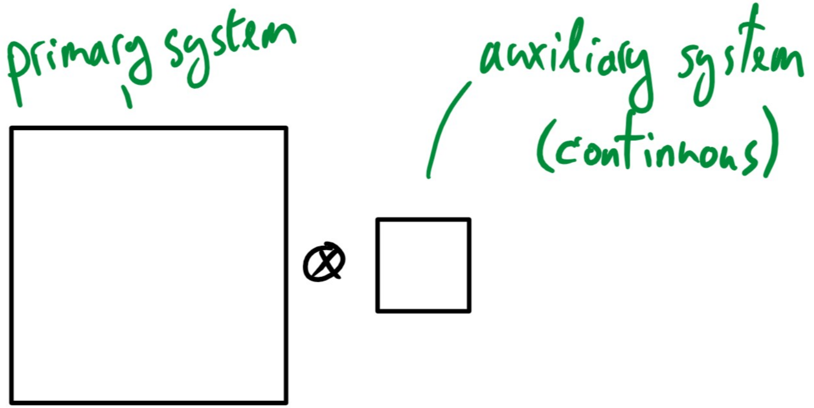

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"Exercise":

$$\left(\mathbb{1} \otimes |N\rangle\langle N|\right) e^{-i\sqrt{2t}H'} \left(|\psi\rangle \otimes |N\rangle\right)$$

$$= \left(e^{-tH^2}|\psi\rangle\right) \otimes |N\rangle \quad \square$$

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! bypass trade-off

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