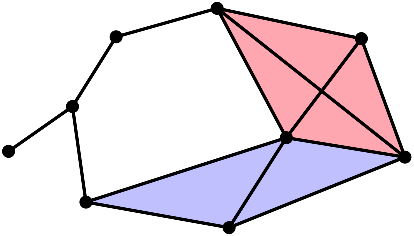
A (SIMPLE) CLASSICAL ALGORITHM FOR ESTIMATING BETTI NUMBERS



Simon Apers
(CNRS & IRIF, Paris)

with **Sander Gribling**, **Dániel Szabó** (IRIF, Paris)
and **Sayantan Sen** (ISI Kolkata)

Phasecraft, February '23

BETTI NUMBERS

A QUANTUM ALGORITHM

A CLASSICAL ALGORITHM

(abstract) simplicial complex

= downward closed set system $K \subseteq 2^{[n]}$

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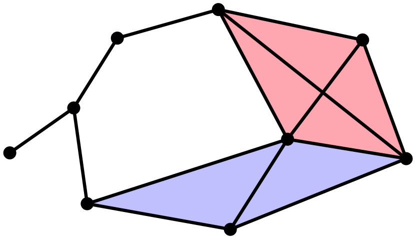
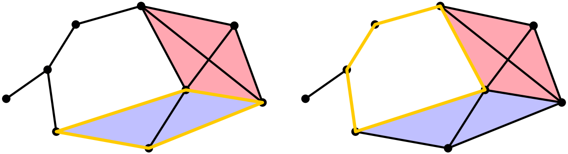
e.g., clique complex



a k-dimensional hole

a k-dimensional hole

has no boundary



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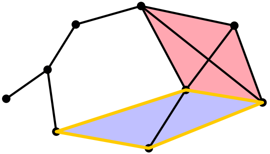
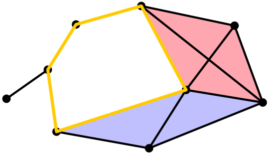
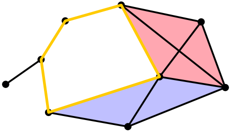




and **is** not a boundary



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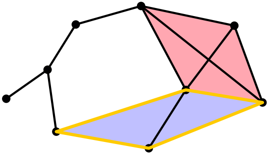

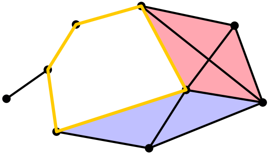

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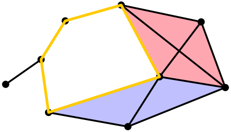

k-th Betti number β_k = # k-dimensional holes

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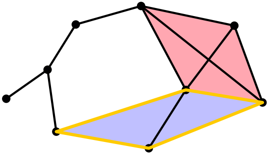

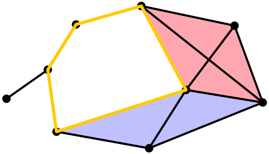

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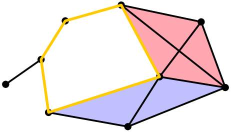

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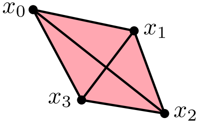
relevant notion in topological data analysis algebraically:

algebraically:

k-face $\quad |S\rangle = |[x_0, x_1, \ldots, x_k]\rangle$



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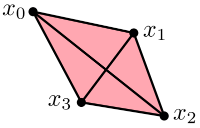
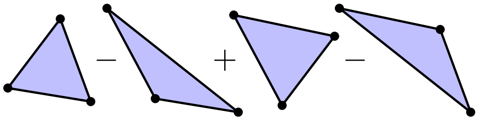


boundary operator $\quad \partial_k |S\rangle = \sum_{\ell=0}^{k} (-1)^\ell |S \backslash \{x_\ell\}\rangle$



a k-dimensional hole

$$|\psi\rangle = \sum \alpha_S \, |S\rangle$$

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$$|\psi\rangle \in \ker(\partial_k) \setminus \operatorname{im}(\partial_{k+1})$$

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Betti number $\beta_k = \dim\left(\ker(\partial_k) \backslash \operatorname{im}(\partial_{k+1})\right)$

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with k-th combinatorial Laplacian

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clique complexes: exponential in input size!

faster algorithms?

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QMA1-hard to multiplicatively approximate
Betti number of clique complex
[Crichigno-Kohler ’22]

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additive approximation $\beta_k/d_k \pm \varepsilon$ in time $\mathrm{poly}(n)$?

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Hermitian matrix A
dimension $d \leq n!$
$\mathrm{poly}(n)$-sparse
nonzero eigenvalues in $[\gamma, 1]$

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? $\frac{\dim(\ker(A))}{d} \pm \varepsilon$ **?**

sampling access to A:
random element from domain,
query neighbors

BETTI NUMBERS

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in $\mathrm{poly}(n, 1/\gamma, 1/\varepsilon)$

A CLASSICAL ALGORITHM

2 quantum techniques:

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• **quantum phase estimation**

for a unitary $U = \sum_\ell e^{i\theta_\ell}|v_\ell\rangle\langle v_\ell|$, maps

$$\sum_\ell \alpha_\ell |v_\ell\rangle|0\rangle \xrightarrow{\text{QPE}} \sum_\ell \alpha_\ell |v_\ell\rangle|\tilde{\theta}_\ell\rangle$$

with $|\tilde{\theta}_\ell - \theta_\ell| \leq \varepsilon$ using $1/\varepsilon$ calls to U

quantum algorithm:

[Lloyd-Garnerone-Zanardi ’14]

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2. apply quantum phase estimation on $|S\rangle$ and U with precision γ

$$\sum_{\ell} \alpha_\ell^S \, |v_\ell\rangle \, |0\rangle \longrightarrow \sum_{\ell} \alpha_\ell^S \, |v_\ell\rangle \, |\tilde{\theta}_\ell\rangle$$

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3. measure phase register: "0" with probability

$$\mathbb{E}\| \Pi_0 |S\rangle \|^2 = \mathbb{E} \sum_{\ell:\tilde{\theta}_\ell = 0} |\alpha_\ell^S|^2 = \beta_k / d_k$$

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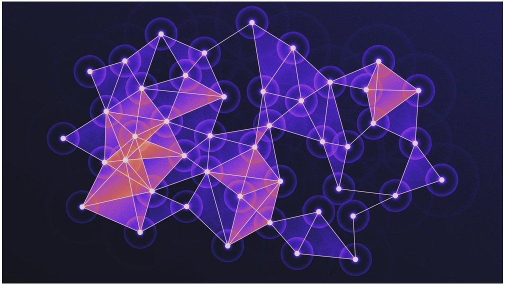
estimate $\dfrac{\beta_k}{d_k} \pm \varepsilon$ in time $\mathrm{poly}(n, 1/\gamma, 1/\varepsilon)$

Quanta magazine

After a Quantum Clobbering, One Approach Survives Unscathed



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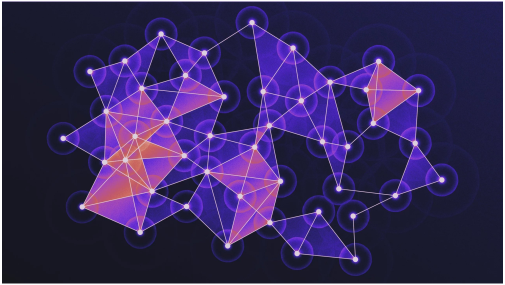

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(clique complex with large gap and many holes)

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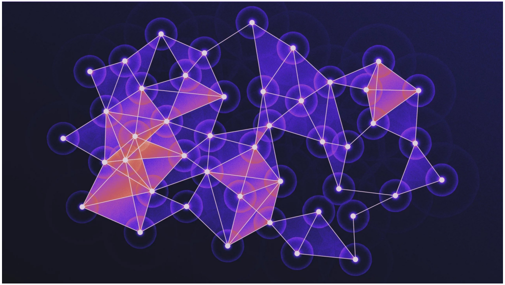

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Betti numbers

A quantum algorithm

in $\mathrm{poly}(n, 1/\gamma, 1/\varepsilon)$

A classical algorithm

in $\mathrm{poly}(n^{\frac{1}{\gamma}\log\frac{1}{\varepsilon}})$

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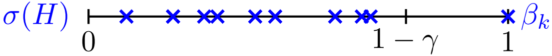
$\rightarrow$ unbiased estimator, variance $\leq \|H\|_1^{2r}$

Betti numbers via PIMC:

$$H = I - \Delta_k/\lambda_{\max}$$



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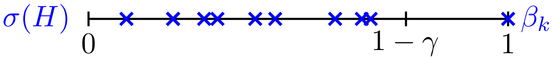

so that

$$\mathrm{Tr}(H^r) = \beta_k \pm \varepsilon d_k$$

if $r \geq \frac{1}{\gamma} \log \frac{1}{\varepsilon}$

Now note

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$$\downarrow$$

estimate $\beta_k/d_k \pm \varepsilon$ in time

$$\operatorname{poly}(n, r, \|H\|_1^r, 1/\varepsilon) \in \operatorname{poly}\left(n^{\frac{1}{\gamma}\log\frac{1}{\varepsilon}}\right)$$

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$$n^{\frac{1}{\gamma}\log\frac{1}{\varepsilon}} \quad\longrightarrow\quad n^{\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}}$$

- better bound on $\|H\|_1 \in O(n/k)$ for clique complexes

$$n^{\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}} \quad\longrightarrow\quad \left(\frac{n}{\lambda_{\max}}\right)^{\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}} \leq \left(\frac{n}{k}\right)^{\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}}$$

Summary and perspective

Betti number estimation in

$\mathrm{poly}(n, 1/\gamma, 1/\varepsilon)$ (quantum)

$\mathrm{poly}(n^{\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}})$ or $\mathrm{poly}(2^{\frac{1}{\sqrt{\gamma}}\log\frac{1}{\varepsilon}})$ (classical)

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[Cade-Crichigno '21]

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- query complexity?

"Betti numbers are testable"
[Elek '10]

- run classical algorithm?

find interesting regimes?

Related works

"Dequantizing the Quantum Singular Value Transformation"

by S. Gharibian and F. Le Gall, '21

related ideas

exact path calculation:
space $\mathrm{poly}(n) \to n^{\frac{1}{\gamma}\log\frac{1}{\varepsilon}}$

our PIMC can exploit small $\|H\|_1$:
$n^{\frac{1}{\gamma}\log\frac{1}{\varepsilon}} \to 2^{\frac{1}{\gamma}\log\frac{1}{\varepsilon}}$ for clique complexes

Related works

"Quantifying Quantum Advantage in Topological Data Analysis"

by D. Berry, Y. Su, C. Gyurik, R. King, J. Basso, A. Del Toro Barba, A. Rajput, N. Wiebe, V. Dunjko and R. Babbush, '22

similar PIMC approach
based on (Trotterized) imaginary time evolution

more complex distribution over paths
to minimize variance

runtime exponential in k and $1/\varepsilon$

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for *bounded degree* complexes
(so $k \in O(1)$)

additive ε-approximation of β_k/d_k
with $f(\varepsilon)$ queries

no gap assumptions