# Directed st-connectivity with few paths is in Quantum Logspace

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INSTITUT
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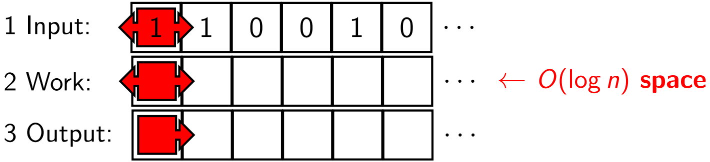
FONDAMENTALE

- **L** = deterministic logspace



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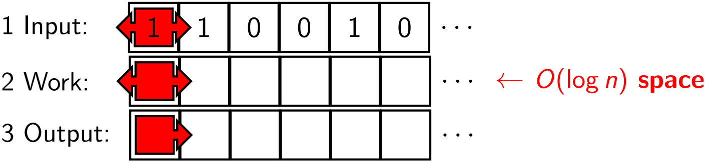
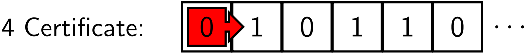



- **NL** = non-deterministic logspace



# Complexity Classes

- **L** = deterministic logspace

  1 Input: 1 1 0 0 1 0 ⋯

  2 Work: ⋯ ← $O(\log n)$ **space**

  3 Output: ⋯

- **NL** = non-deterministic logspace

  4 Certificate: 0 1 0 1 1 0 ⋯

- **BPL** = bounded-error randomized logspace

  4' Coins: 1 0 0 1 0 1 ⋯

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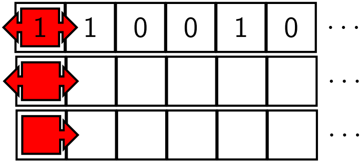


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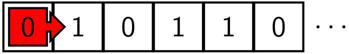


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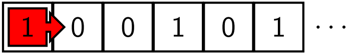
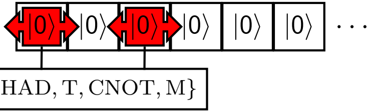



- **BQL** = bounded-error quantum logspace

4'' Quantum: ← $O(\log n)$ **space**



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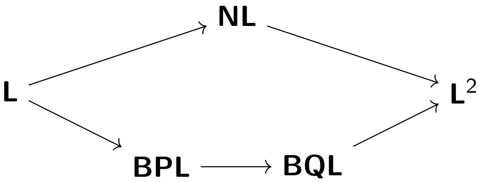

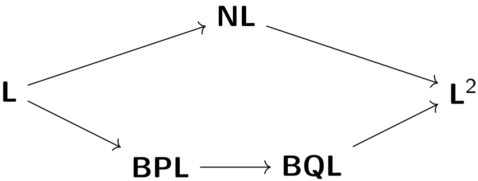


- **NL** $\rightarrow$ $\mathbf{L}^2$ [Savitch '70]

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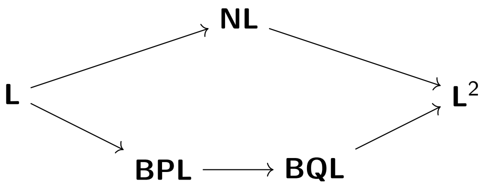


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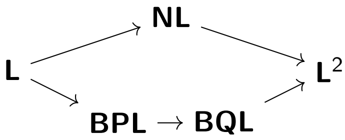
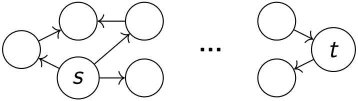
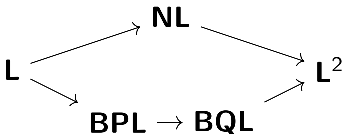



- STCON: *st*-connectivity on a *directed* graph



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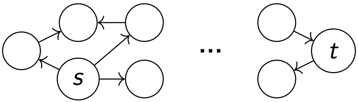
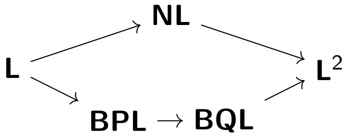



**NL**-complete

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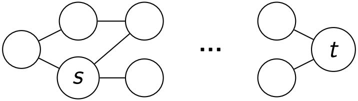

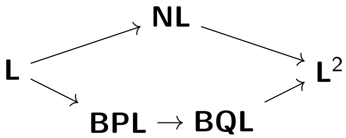


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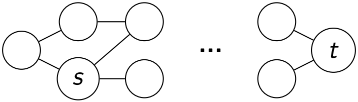
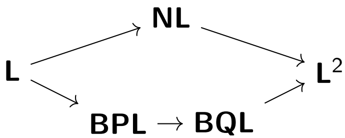



**L**-hard [Cook-McKenzie '87]

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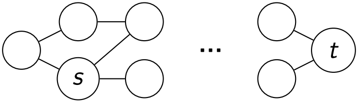


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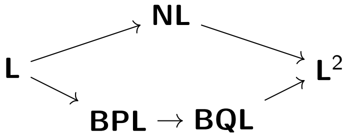


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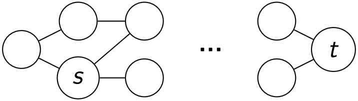


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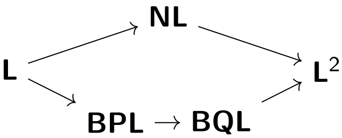
$\in$ **L** [Reingold '05] (derandomization)

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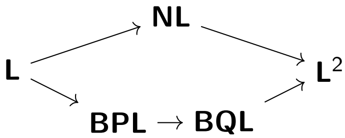


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given $A \in \mathbb{C}^{n\times n}$ with $\|A\|, \|A^{-1}\| \leq \mathrm{poly}(n)$,
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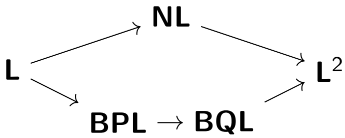


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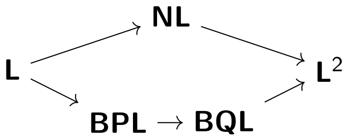


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**BQL**-complete [Fefferman-Lin '16]

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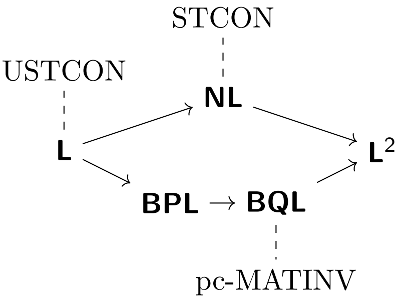


## Motivating question

*quantum advantage for STCON?*

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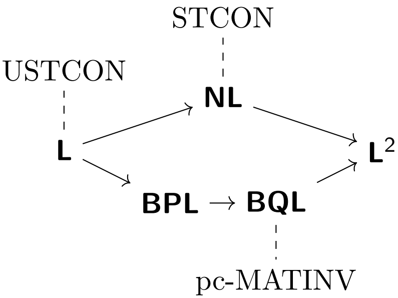


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- Restricted version of STCON in **BQL**

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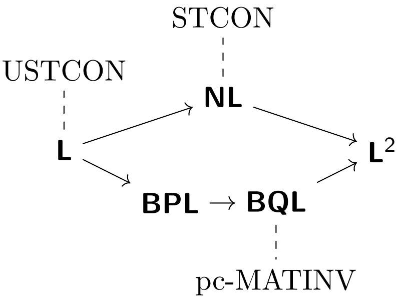


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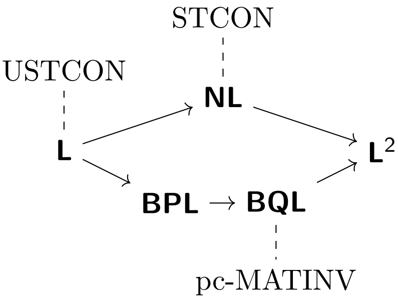


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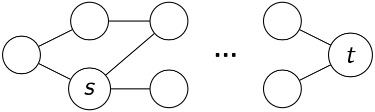
## Main results

- Restricted version of STCON in **BQL** (and $1000 for putting it in **BPL**)
- First language in **BQL** not known to be in **L**

# Random walks

## Undirected graph



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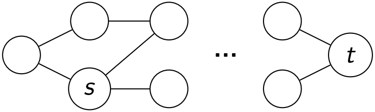


hitting time $O(n^3)$

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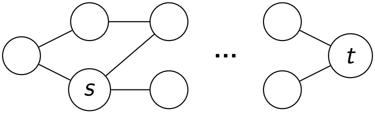


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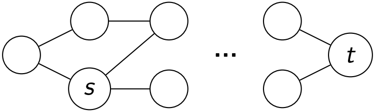


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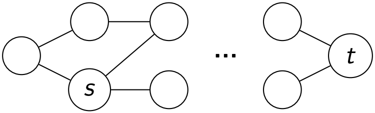


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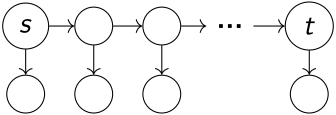

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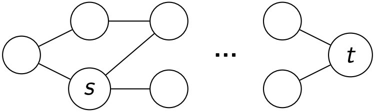


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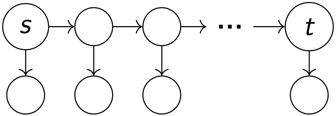


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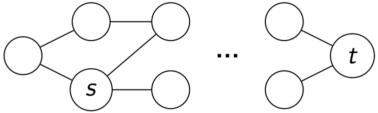


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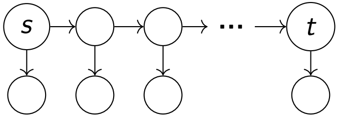


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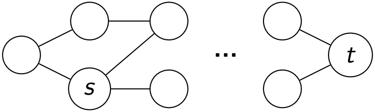


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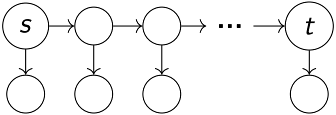


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Let $N(i, j) := \#\text{paths}(i, j)$.

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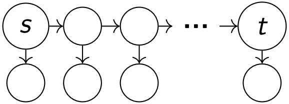
## Few paths in **BQL**

If $N(i,j) \leq \mathrm{poly}(n)\ \forall i,j$, we can return $N(s,t)$ in **BQL**.



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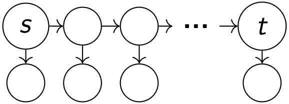


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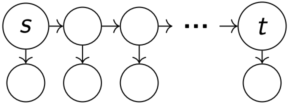


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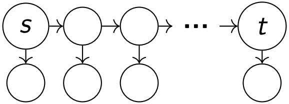


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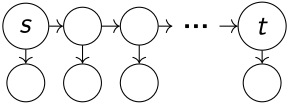


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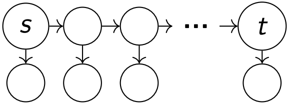


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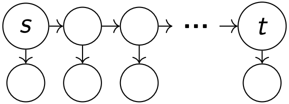


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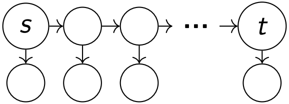


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$\Rightarrow \mathcal{L}$ is poly-conditioned iff $\forall i,j : N(i,j) \leq \mathrm{poly}(n)$.

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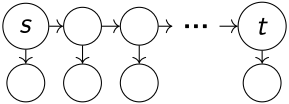


### Corollary

$\mathrm{STCON}_{\mathrm{sf}} := \{\langle G, s, t, 1^k \rangle \mid \forall i, j : N(i, j) \leq k \text{ and } N(s, t) \geq 1\} \in$ **BQL**.

# A language

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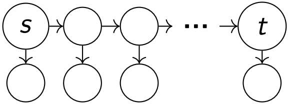


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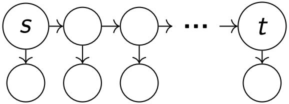


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For comparison:

- pc-MATINV [Ta-Shma '13]
- pc-MATPOW [Fefferman-Remscrim '20]
- quantum state testing [Le Gall-Liu-Wang '23]

all **promise problems**

# A weaker promise

## Few outgoing/incoming paths in **BQL**

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**Proof.** Consider $\mathcal{L} := I - A = \sum_j \sigma_j \left|u_j\right\rangle \left\langle v_j\right|$. Pick $\kappa = 1/\mathrm{poly}(n)$.

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**Few outgoing/incoming paths in BQL**

If $N(s,\cdot)$, $N(\cdot,t) \leq \mathrm{poly}(n)$, we can return $N(s,t)$ in **BQL**.

# Comparison

## Few outgoing/incoming paths in **BQL**

If $N(s,\cdot)$, $N(\cdot,t) \leq \mathrm{poly}(n)$, we can return $N(s,t)$ in **BQL**.

## Few outgoing paths in $\mathbf{DSPACE}(o(\log^2 n))$

If $N(s,\cdot) \leq \mathrm{poly}(n)$, can decide STCON in $\mathbf{DSPACE}\left(\frac{\log^2(n)}{\log\log(n)}\right)$.

[Allender-Lange '98], [Garvin-Stolee-Tewari-Vinodchandran '11]

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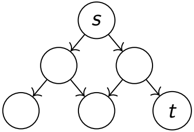
## Few outgoing paths in $\mathbf{SC}^2$

If $N(s, \cdot) \leq \mathrm{poly}(n)$, can decide STCON in $\mathrm{poly}(n)$ time and $O(\log^2 n)$ space.

[Lange '97], [Garvin-Stolee-Tewari-Vinodchandran '11]

**!** can connect to existing complexity classes: $\langle G, s, t \rangle$ is called ...




unambiguous

$:\Leftrightarrow N(s,t) \leq 1$

few unamb.

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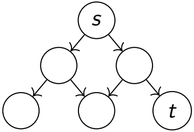


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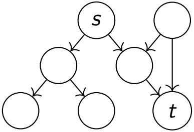


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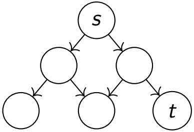


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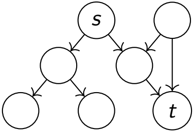


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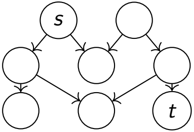


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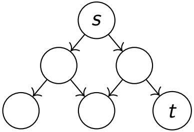


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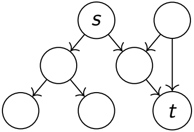


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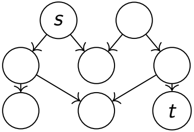


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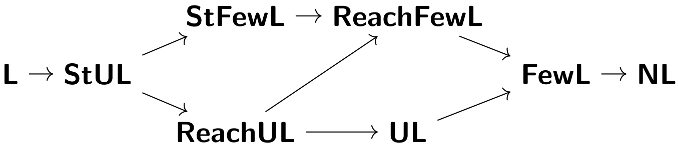
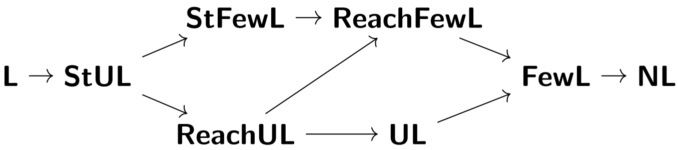
$\Rightarrow$ corresponding complexity classes:



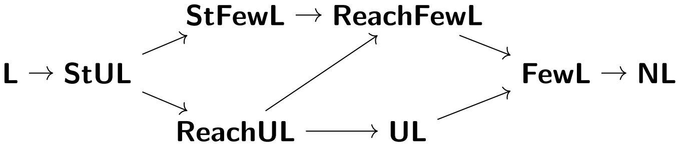
# Unambiguity & Fewness




**Corollary:**

**StFewL** $\subseteq$ **BQL**

**Further Results:**

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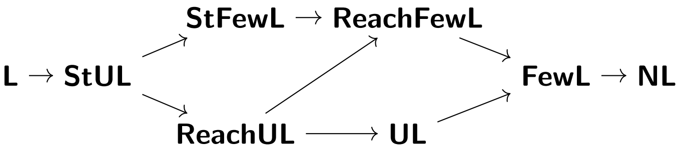
1. [AL98]: **ReachUL** $\subseteq \mathrm{DSPACE}(\frac{\log^2 n}{\log \log n})$

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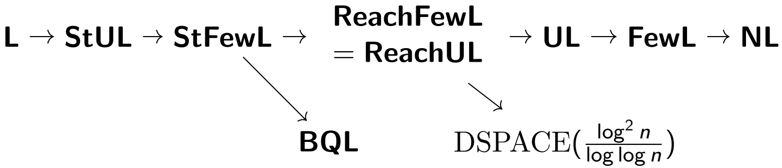
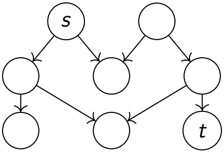
1. [AL98]: **ReachUL** $\subseteq \mathrm{DSPACE}(\frac{\log^2 n}{\log\log n})$
2. [GSTV11]: **ReachUL** $=$ **ReachFewL**



# Open Questions

## \$1000 reward for a dequantization [All23]

STCON on StU graphs in DSPACE$(o(\log^2 n/\log\log n))$?




$\forall i, j : N(i, j) \leq 1$

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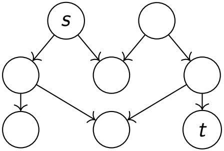


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- **BQL**-hardness of STCON-variant?

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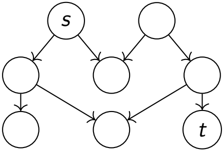


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- **BQL**-hardness of STCON-variant?
- Further investigation of quantum algorithms on directed graphs!

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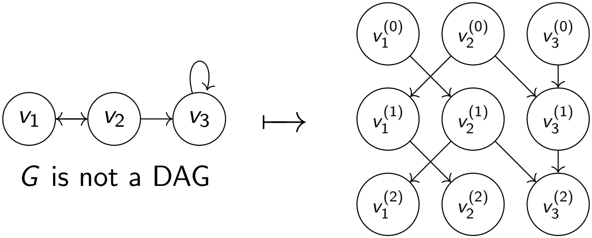
## DAG reduction




$G$ is not a DAG

$G'$ is a DAG