Lecture 4: Transducers and quantum Las Vegas

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In this lecture we introduce a new tool in quantum algorithms called *transducers*. We then use this tool to prove a remarkable statement about quantum query complexity.

1 Transducers

Transducers were recently introduced by Belovs, Jeffery and Yolcu [BJY24].

Definition 1. *Consider a unitary U acting on a direct-sum space $A \oplus B$. Let $|\xi\rangle, |\tau\rangle \in A$ be normalized, and $|v\rangle \in B$ potentially unnormalized. If*

$$|\xi\rangle \oplus |v\rangle \xrightarrow{U} |\tau\rangle \oplus |v\rangle$$

then U is said to transduce *$|\xi\rangle$ into $|\tau\rangle$ with* catalyst *$|v\rangle$. We write $|\xi\rangle \overset{U}{\rightsquigarrow} |\tau\rangle$ and call $W = \| \, |v\rangle \, \|^2$ the* transduction complexity.

In fact, for every $|\xi\rangle$ there exists a unique $|\tau\rangle$ and a (potentially non-unique) $|v\rangle$ such that $U(|\xi\rangle \oplus |v\rangle) = |\tau\rangle \oplus |v\rangle$. The interest in transducers is the following theorem.

Theorem 1 ([BJY24])**.** *Let $|\xi\rangle \overset{U}{\rightsquigarrow} |\tau\rangle$ with transduction complexity W. There exists a quantum algorithm mapping $|\xi\rangle$ to a state ε-close to $|\tau\rangle$ using $\kappa \in O(W/\varepsilon^2)$ calls to U and $O(\kappa)$ other gates.*

Proof. First we will reduce the catalyst complexity by effectively calling the transducer multiple times. To this end, we attach an additional clock register and consider the initial state $|\hat{\xi}\rangle = |\xi\rangle \frac{1}{\sqrt{\kappa}} \sum_{k=0}^{\kappa-1} |k\rangle$. We will (later) construct a unitary V so that

$$|\hat{\xi}\rangle \oplus \frac{1}{\sqrt{\kappa}} |v\rangle |0\rangle \xrightarrow{V} |\hat{\tau}\rangle \oplus \frac{1}{\sqrt{\kappa}} |v\rangle |0\rangle ,$$

with $|\hat{\tau}\rangle = |\tau\rangle \frac{1}{\sqrt{\kappa}} \sum_{k=0}^{\kappa-1} |k\rangle$. This implies that V transduces $|\hat{\xi}\rangle$ into $|\hat{\tau}\rangle$ with reduced transduction complexity W/κ.

This suffices to prove the theorem because we can pick $\kappa \in O(W/\varepsilon^2)$ so that $\left\| \frac{1}{\sqrt{\kappa}} |v\rangle |0\rangle \right\| \in O(\varepsilon)$ and in particular

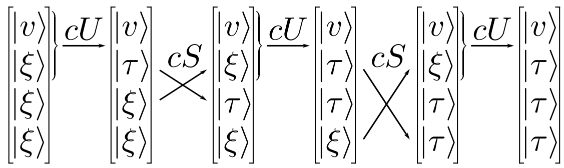
$$|\hat{\xi}\rangle \xrightarrow{V} |\hat{\tau}\rangle + (I - V)\frac{1}{\sqrt{\kappa}} |v\rangle |0\rangle = |\hat{\tau}\rangle + O(\varepsilon).$$

I.e., once the catalyst has norm $O(\varepsilon)$ we can just drop it. The resulting algorithm then applies V to $|\hat{\xi}\rangle$ and returns an ε-approximation of $|\hat{\tau}\rangle$.

It remains to construct the unitary V, which we build up by transducing $|\hat{\xi}\rangle$ one part at a time, each time reusing the same catalyst. For the case $\kappa = 3$ this corresponds to the following scheme (suppressing the factor $1/\sqrt{\kappa}$):

$$\begin{aligned}
|v\rangle|0\rangle + |\xi\rangle|0\rangle + |\xi\rangle|1\rangle + |\xi\rangle|2\rangle &\xrightarrow{cU} |v\rangle|0\rangle + |\tau\rangle|0\rangle + |\xi\rangle|1\rangle + |\xi\rangle|2\rangle \\
&\xrightarrow{\mathrm{cS}_{0,1}} |v\rangle|0\rangle + |\xi\rangle|0\rangle + |\tau\rangle|1\rangle + |\xi\rangle|2\rangle \\
&\xrightarrow{cU} |v\rangle|0\rangle + |\tau\rangle|0\rangle + |\tau\rangle|1\rangle + |\xi\rangle|2\rangle \\
&\xrightarrow{\mathrm{cS}_{0,2}} |v\rangle|0\rangle + |\xi\rangle|0\rangle + |\tau\rangle|1\rangle + |\tau\rangle|2\rangle \\
&\xrightarrow{cU} |v\rangle|0\rangle + |\tau\rangle|0\rangle + |\tau\rangle|1\rangle + |\tau\rangle|2\rangle,
\end{aligned}$$

where cU is U controlled on ancilla register 0, and $\mathrm{cS}_{0,k}$ swaps ancilla registers 0 and k controlled on not swapping $|v\rangle$ (i.e., it is controlled to act on A and not on the catalyst space B). Schematically, this corresponds to the following:



For general κ, the resulting unitary is

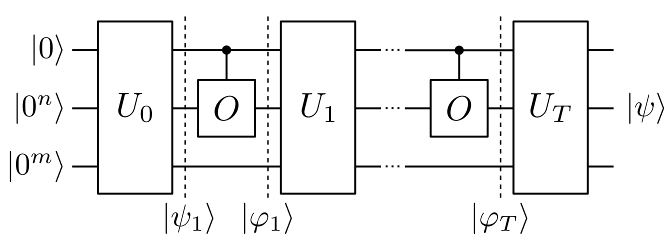
$$V = cU \cdot \mathrm{cS}_{0,\kappa} \cdot cU \cdot \mathrm{cS}_{0,\kappa-1} \cdot \ \ldots\ \cdot \mathrm{cS}_{0,1} \cdot cU.$$

It effectively makes κ calls to U and maps $|\hat{\xi}\rangle + \frac{1}{\sqrt{\kappa}}|v\rangle|0\rangle \xrightarrow{V} |\hat{\tau}\rangle + \frac{1}{\sqrt{\kappa}}|v\rangle|0\rangle$. □

Transducers form a sort of "idealized" proxy for actual quantum algorithms, and this allows them to have many useful properties. E.g., there exist zero-error transducers for many algorithmic tasks. They also compose particularly nicely, and it has been suggested that these are the more natural model for designing quantum algorithms (as compared to the circuit model) [Jef25].

2 Quantum Las Vegas complexity

We will use transducers to prove a remarkable (and useful) fact about quantum query algorithms. Let the following circuit represent a general T-query quantum algorithm mapping input $|0\rangle$ to output $|\psi\rangle$, with O the (controlled) phase oracle and the U_i's independent of O.



We use $|\psi_k\rangle = |\psi_k^0\rangle + |\psi_k^1\rangle$ and $|\varphi_k\rangle = |\varphi_k^0\rangle + |\varphi_k^1\rangle$ to denote the state right before and after the k-th query, with $|\psi_k^i\rangle = (|i\rangle\langle i| \otimes I)|\psi_k\rangle$ and $|\varphi_k^i\rangle = (|i\rangle\langle i| \otimes I)|\varphi_k\rangle$ for $i \in \{0,1\}$ their (unnormalized) components. We can then write

$$\begin{aligned}
|\psi_k\rangle &= |\psi_k^0\rangle + |\psi_k^1\rangle \\
&\overset{cO}{\mapsto} |\varphi_k\rangle = |\varphi_k^0\rangle + |\varphi_k^1\rangle,
\end{aligned}$$

where clearly $|\varphi_k^1\rangle = O\,|\psi_k^1\rangle$. Intuitively, the "effective" query weight for the k-th query is then the squared norm of the k-th "query state", $\|\,|\psi_k^1\rangle\,\|^2$. Following this intuition, the *quantum Las Vegas complexity* of a quantum query algorithm is defined as

$$Q_{LV} = \sum_{k=1}^{T} \|\,|\psi_k^1\rangle\,\|^2.$$

This notion parallels the Las Vegas complexity of a randomized algorithm, which counts its number of queries *in expectation.* While bounded by the usual quantum query complexity, $Q_{LV} \leq T$, the quantum Las Vegas complexity can be much lower than T. Nonetheless, we can use transducers to prove the following theorem.

Theorem 2 ([BY23, BJY24]). *Any algorithm with quantum Las Vegas complexity Q_{LV} can be turned into a bounded-error algorithm that makes $O(Q_{LV})$ quantum queries.*

Proof. We will construct a unitary V that makes a single query to the oracle O and that transduces the algorithm input $|0\rangle$ into the output $|\psi\rangle$ with transduction complexity Q_{LV}. By Theorem 1 this implies the existence of a quantum algorithm mapping $|0\rangle$ to a state close to $|\psi\rangle$ while making $O(Q_{LV})$ calls to U, and so in particular $O(Q_{LV})$ quantum queries to O.

The construction of V is a variation on the earlier construction. After appending an ancilla clock register, we will design V so that

$$|0\rangle\,|0\rangle + \sum_{k=1}^{T} |\psi_k^1\rangle\,|k\rangle \xrightarrow{V} |\psi\rangle\,|0\rangle + \sum_{k=1}^{T} |\psi_k^1\rangle\,|k\rangle\,.$$

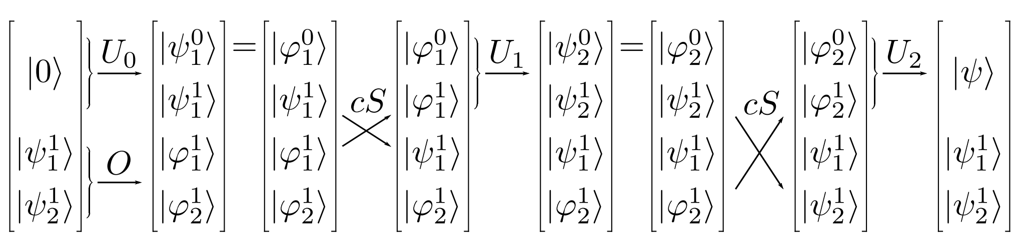
This shows that $|0\rangle\,|0\rangle \overset{V}{\rightsquigarrow} |\psi\rangle\,|0\rangle$ with the superposition of all query states $\sum_{k=1}^{T} |\psi_k^1\rangle\,|k\rangle$ being the catalyst, and transduction complexity

$$\left\| \sum_{k=1}^{T} |\psi_k^1\rangle\,|k\rangle \right\|^2 = Q_{LV}.$$

We demonstrate the construction for the case $T = 2$. Key idea is to apply the oracle once to the superposition of all query states:

$$|0\rangle\,|0\rangle + |\psi_1^1\rangle\,|1\rangle + |\psi_2^1\rangle\,|2\rangle \xrightarrow{cO} |0\rangle\,|0\rangle + |\phi_1^1\rangle\,|1\rangle + |\phi_2^1\rangle\,|2\rangle\,.$$

Following this by elementary swaps and calls to the U_i's, we can convert the resulting state into the goal state $|\psi\rangle\,|0\rangle + |\psi_1^1\rangle\,|1\rangle + |\psi_2^1\rangle\,|2\rangle$. Schematically, we do the following:



As claimed, the resulting unitary V transduces $|0\rangle$ into $|\psi\rangle$ while making a single oracle call. □

References

[BJY24] Aleksandrs Belovs, Stacey Jeffery, and Duyal Yolcu. Taming quantum time complexity. *Quantum*, 8:1444, 2024.

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