MPRI: Quantum Algorithms and Complexity (QUALCO) **2025 - 2026**

Lecture 2: Quantum linear algebra

Lecturer: Simon Apers (apers@irif.fr)

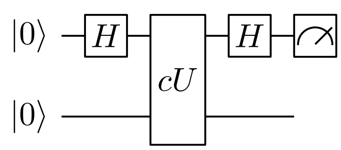
In this lecture we touch on the topic of "quantum linear algebra". Broadly construed, this is the use of quantum algorithms to do linear algebraic operations such as summing matrices or solving linear systems.

1 Linear Combination of Unitaries

Let cU denote the controlled unitary

$$cU = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1,$$

and recall the quantum Hadamard test.

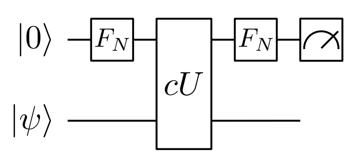


Exercise 1.
- *What is the output of this circuit if we postselect on measurement outcome "0"?*
- *What is the probability of obtaining outcome "0"?*

We can generalize this to more unitaries $U_0, U_1, \ldots, U_{N-1}$ for $N = 2^n$ by defining the controlled-unitary

$$cU = \sum_{k=0}^{N-1} |k\rangle\langle k| \otimes U_k$$

and invoking the following circuit, where F_N is the N-dimensional Fourier transform:



Exercise 2.
- *What is the output of this circuit if we postselect on measurement outcome "0"?*
- *What is the probability of obtaining outcome "0"?*

Generalizing this even further, consider a set of nonnegative coefficients $c_1, c_2, \ldots, c_N$ satisfying $\sum_k c_k = 1$ and define the "select unitary" U_{sel} as any unitary satisfying

$$U_{\text{sel}} |0\rangle = \sum_{k=0}^{N-1} \sqrt{c_k} |k\rangle .$$

Now consider the following circuit, which implements the so-called "linear combination of unitaries" (LCU) technique:

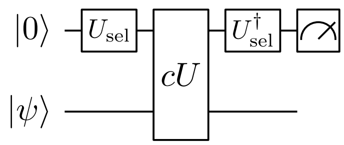


Figure 1: LCU circuit.

Exercise 3. • *What is the output of this circuit if we postselect on measurement outcome "0"?*

2 Quantum linear system solving

Consider a linear system $Ax = b$ for some invertible, Hermitian matrix $A \in \mathbb{C}^{N\times N}$. Given appropriate query access to A and b, we wish to compute the solution $x = A^{-1}b$. In a famous work, Harrow, Hassidim and Lloyd [HHL09] proposed a quantum algorithm that returns a *quantum state* $|x\rangle$ encoding the solution in time $\mathrm{poly}(\kappa) \cdot \mathrm{polylog}(N)$. Here κ is the condition number of A, defined by the ratio of the largest over the smallest eigenvalue of A in magnitude. If $\kappa \in O(\mathrm{polylog}(N))$ (i.e., A is "well-conditioned") then this is exponentially faster than the usual classical algorithms for matrix inversion, which take time $\mathrm{poly}(N)$.

While there are different versions of the HHL algorithm, we will describe the algorithm in [CKS17] that combines Hamiltonian simulation with the LCU technique. At its core is the existence of an (approximate) Fourier expansion of the inverse function $1/x$ (valid for the range $1/\kappa \leq |x| \leq 1$) of the form

$$\frac{1}{x} \approx \kappa \sum_{k=0}^{O(\kappa)} c_k e^{ikx}, \tag{1}$$

where for simplicity we assume that $c_k \geq 0$ and $\sum_k c_k = 1$. This implies that we can also rewrite the matrix inverse

$$A^{-1} \approx \kappa \sum_{k=0}^{O(\kappa)} c_k e^{ikA},$$

and so A^{-1} can be approximated by a linear combination of (unitary!) matrix exponentials. This leads to an algorithm by defining the c_k's in U_{sel} to be those in (1), and picking the unitaries

$$U_k = e^{ikA}.$$

Notice that we can implement these unitaries using a quantum algorithm for Hamiltonian simulation. The algorithm is then given by the LCU circuit from Fig. 1.

Exercise 4.

- *What is the output of the circuit, when applied to the state $|0\rangle\,|b\rangle$, and postselected on outcome "0"?*
- *Reason about the complexity of the resulting algorithm.*

References

[CKS17] Andrew M. Childs, Robin Kothari, and Rolando D. Somma. Quantum algorithm for systems of linear equations with exponentially improved dependence on precision. *SIAM Journal on Computing*, 46(6):1920–1950, 2017.

[HHL09] Aram W. Harrow, Avinatan Hassidim, and Seth Lloyd. Quantum algorithm for linear systems of equations. *Physical review letters*, 103(15):150502, 2009.